

EE 470 - Signals and Systems

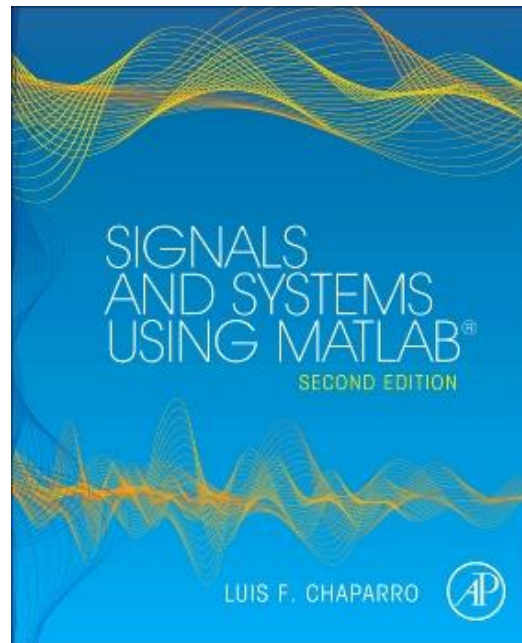
4. The Fourier Transform

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Textbook

Luis Chapparo, Signals and Systems Using Matlab, 2nd ed., Academic Press, 2015.



Fourier Transform Definition



$$x(t) \Leftrightarrow X(\Omega)$$

where the signal $x(t)$ is transformed into a function $X(\Omega)$ in the frequency domain by the

Fourier transform:
$$X(\Omega) = \int_{-\infty}^{\infty} x(t) e^{-j\Omega t} dt$$

while $X(\Omega)$ is transformed into a signal $x(t)$ in the time domain by the

Inverse Fourier transform:
$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\Omega) e^{j\Omega t} d\Omega$$



Existence of Fourier Transform

- The Fourier transform of a signal $x(t)$ exists (i.e., we can calculate its Fourier transform via this integral) provided that:
 - $x(t)$ is absolutely integrable or the area under $|x(t)|$ is finite
 - $x(t)$ has only a finite number of discontinuities as well as maxima and minima
- These conditions are “sufficient” not “necessary”

Fourier Transforms from Laplace Transforms

- If the region of convergence (ROC) of the Laplace transform $X(s)$ contains the $j\Omega$ axis, so that $X(s)$ can be defined for $s=j\Omega$, then:

$$\begin{aligned}\mathcal{F}[x(t)] &= \mathcal{L}[x(t)]|_{s=j\Omega} = \int_{-\infty}^{\infty} x(t)e^{-j\Omega t} dt \\ &= X(s) \big|_{s=j\Omega}\end{aligned}$$

Fourier Transforms from Laplace Transforms - Example

- Discuss whether it is possible to obtain the Fourier transform of the following signals using their Laplace transforms:

$$(a) \quad x_1(t) = u(t)$$

$$(b) \quad x_2(t) = e^{-2t}u(t)$$

- (a) The Laplace transform of $x_1(t)$ is $X_1(s) = 1/s$ with a region of convergence corresponding to the open right s -plane, or $\text{ROC} = \{s = \sigma + j\Omega : \sigma > 0, -\infty < \Omega < \infty\}$, which does not include the $j\Omega$ axis, so the Laplace transform cannot be used to find the Fourier transform of $x_1(t)$.
- (b) The signal $x_2(t)$ has as Laplace transform $X_2(s) = 1/(s + 2)$ with a region of convergence $\text{ROC} = \{s = \sigma + j\Omega : \sigma > -2, -\infty < \Omega < \infty\}$ containing the $j\Omega$ axis. Then the Fourier transform of $x_2(t)$ is

$$X_2(\Omega) = \frac{1}{s + 2} \Big|_{s=j\Omega} = \frac{1}{j\Omega + 2}$$

Table 5.2 Fourier Transform Pairs

	Function of Time	Function of Ω
1	$\delta(t)$	1
2	$\delta(t - \tau)$	$e^{-j\Omega\tau}$
3	$u(t)$	$\frac{1}{j\Omega} + \pi\delta(\Omega)$
4	$u(-t)$	$\frac{-1}{j\Omega} + \pi\delta(\Omega)$
5	$\text{sgn}(t) = 2[u(t) - 0.5]$	$\frac{2}{j\Omega}$
6	$A, -\infty < t < \infty$	$2\pi A\delta(\Omega)$
7	$Ae^{-at}u(t), a > 0$	$\frac{A}{j\Omega + a}$
8	$Ate^{-at}u(t), a > 0$	$\frac{A}{(j\Omega + a)^2}$
9	$e^{-a t }, a > 0$	$\frac{2a}{a^2 + \Omega^2}$
10	$\cos(\Omega_0 t), -\infty < t < \infty$	$\pi[\delta(\Omega - \Omega_0) + \delta(\Omega + \Omega_0)]$
11	$\sin(\Omega_0 t), -\infty < t < \infty$	$-j\pi[\delta(\Omega - \Omega_0) - \delta(\Omega + \Omega_0)]$
12	$A[u(t + \tau) - u(t - \tau)], \tau > 0$	$2A\tau \frac{\sin(\Omega\tau)}{\Omega\tau}$
13	$\frac{\sin(\Omega_0 t)}{\pi t}$	$u(\Omega + \Omega_0) - u(\Omega - \Omega_0)$
14	$x(t) \cos(\Omega_0 t)$	$0.5[X(\Omega - \Omega_0) + X(\Omega + \Omega_0)]$

Table 5.1 Basic Properties of the Fourier Transform

	Time Domain	Frequency Domain
Signals and constants	$x(t), y(t), z(t), \alpha, \beta$	$X(\Omega), Y(\Omega), Z(\Omega)$
Linearity	$\alpha x(t) + \beta y(t)$	$\alpha X(\Omega) + \beta Y(\Omega)$
Expansion/contraction in time	$x(\alpha t), \alpha \neq 0$	$\frac{1}{ \alpha } X\left(\frac{\Omega}{\alpha}\right)$
Reflection	$x(-t)$	$X(-\Omega)$
Parseval's energy relation	$E_x = \int_{-\infty}^{\infty} x(t) ^2 dt$	$E_x = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\Omega) ^2 d\Omega$
Duality	$X(t)$	$2\pi x(-\Omega)$
Time differentiation	$\frac{d^n x(t)}{dt^n}, n \geq 1, \text{ integer}$	$(j\Omega)^n X(\Omega)$
Frequency differentiation	$-jtx(t)$	$\frac{dX(\Omega)}{d\Omega}$
Integration	$\int_{-\infty}^t x(t') dt'$	$\frac{X(\Omega)}{j\Omega} + \pi X(0)\delta(\Omega)$
Time shifting	$x(t - \alpha)$	$e^{-j\alpha\Omega} X(\Omega)$
Frequency shifting	$e^{j\Omega_0 t} x(t)$	$X(\Omega - \Omega_0)$
Modulation	$x(t) \cos(\Omega_c t)$	$0.5[X(\Omega - \Omega_c) + X(\Omega + \Omega_c)]$
Periodic signals	$x(t) = \sum_k X_k e^{jk\Omega_0 t}$	$X(\Omega) = \sum_k 2\pi X_k \delta(\Omega - k\Omega_0)$
Symmetry	$x(t) \text{ real}$	$ X(\Omega) = X(-\Omega) $ $\angle X(\Omega) = -\angle X(-\Omega)$
Convolution in time	$z(t) = [x * y](t)$	$Z(\Omega) = X(\Omega)Y(\Omega)$
Windowing/multiplication	$x(t)y(t)$	$\frac{1}{2\pi} [X * Y](\Omega)$
Cosine transform	$x(t) \text{ even}$	$X(\Omega) = \int_{-\infty}^{\infty} x(t) \cos(\Omega t) dt, \text{ real}$
Sine transform	$x(t) \text{ odd}$	$X(\Omega) = -j \int_{-\infty}^{\infty} x(t) \sin(\Omega t) dt, \text{ imaginary}$

Linearity

- Fourier transform is a linear operator
 - Superposition holds

If $\mathcal{F}[x(t)] = X(\Omega)$ and $\mathcal{F}[y(t)] = Y(\Omega)$, for constants α and β , we have that

$$\begin{aligned}\mathcal{F}[\alpha x(t) + \beta y(t)] &= \alpha \mathcal{F}[x(t)] + \beta \mathcal{F}[y(t)] \\ &= \alpha X(\Omega) + \beta Y(\Omega)\end{aligned}$$

Inverse Proportionality of Time and Frequency

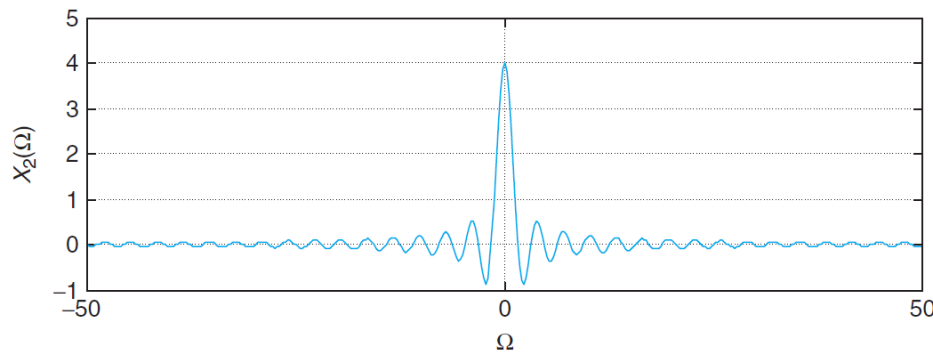
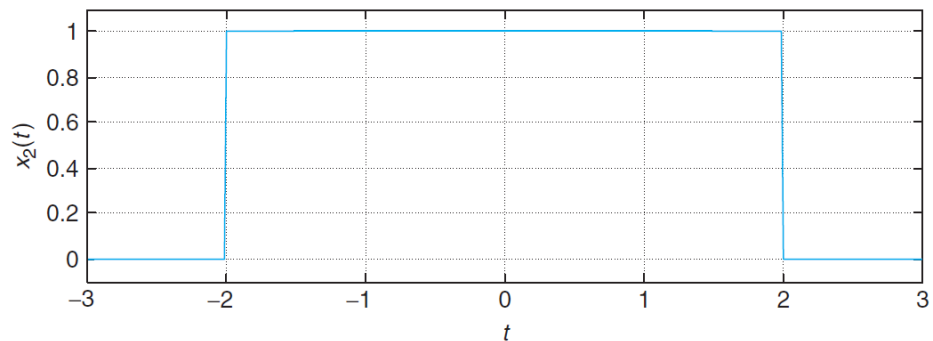
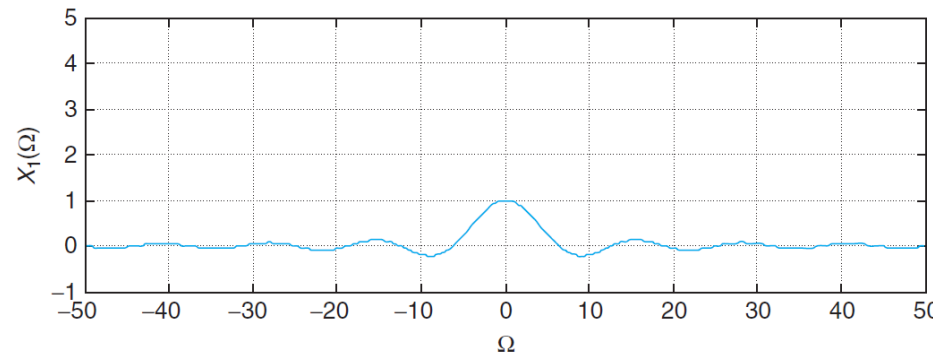
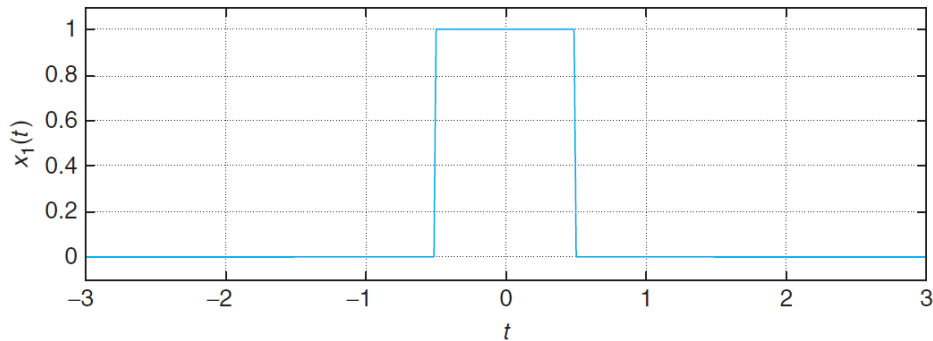
- Support of $X(\Omega)$ is inversely proportional to support of $x(t)$
- If $x(t)$ has a Fourier transform $X(\Omega)$ and $\alpha \neq 0$ is a real number, then $x(\alpha t)$ is:
 - Contracted ($\alpha > 1$),
 - Contracted and reflected ($\alpha < -1$),
 - Expanded ($0 < \alpha < 1$),
 - Expanded and reflected ($-1 < \alpha < 0$), or
 - Simply reflected ($\alpha = -1$)

- Then,

$$x(\alpha t) \Leftrightarrow \frac{1}{|\alpha|} X\left(\frac{\Omega}{\alpha}\right)$$

Inverse Proportionality of Time and Frequency - Example

- Fourier transform of 2 pulses of different width
 - 4-times wider pulse have 4-times narrower Fourier transform



Duality

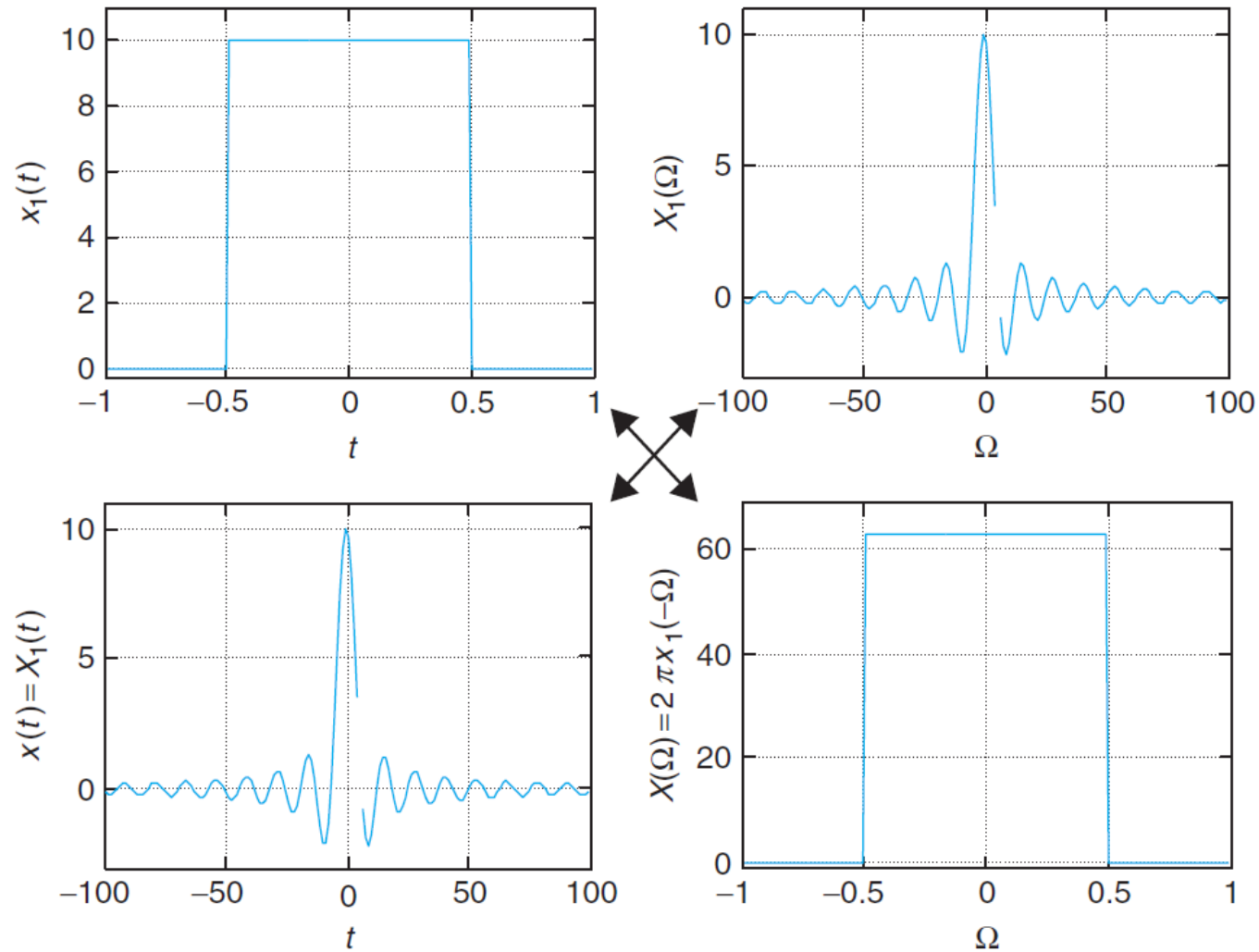
- By interchanging the frequency and the time variables in the definitions of the direct and the inverse Fourier transform similar equations are obtained
- Thus, the direct and the inverse Fourier transforms are termed to be dual

$$x(t) \Leftrightarrow X(\Omega)$$



$$X(t) \Leftrightarrow 2\pi x(-\Omega)$$

Duality: Example



Signal Modulation

- Frequency shift: If $X(\Omega)$ is the Fourier transform of $x(t)$, then we have the pair

$$x(t)e^{j\Omega_0 t} \Leftrightarrow X(\Omega - \Omega_0)$$

- Modulation: The Fourier transform of the modulated signal

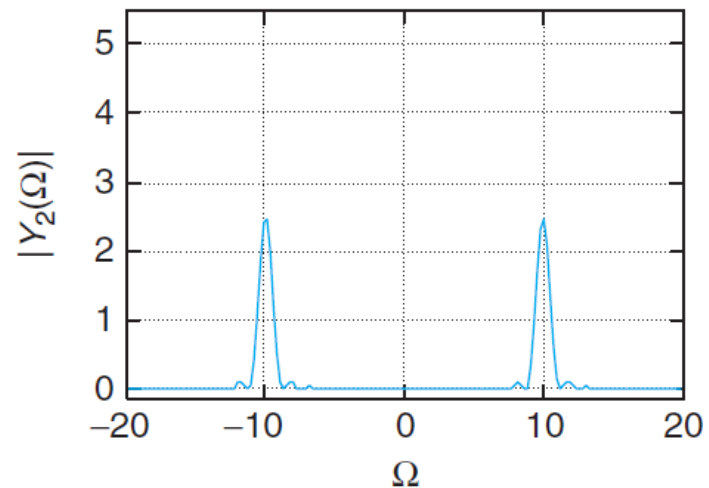
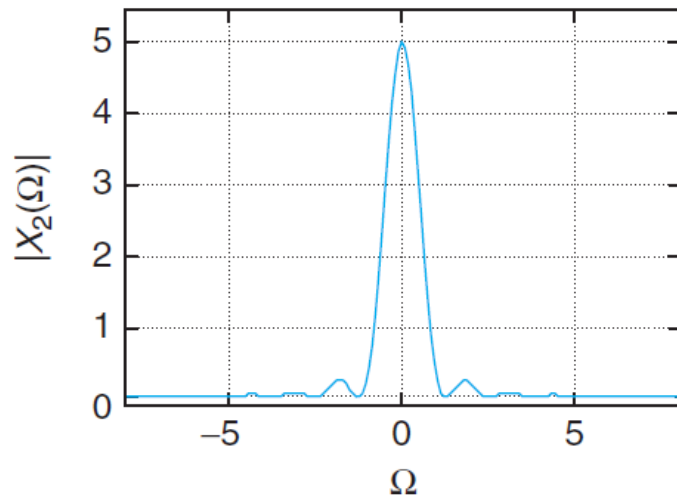
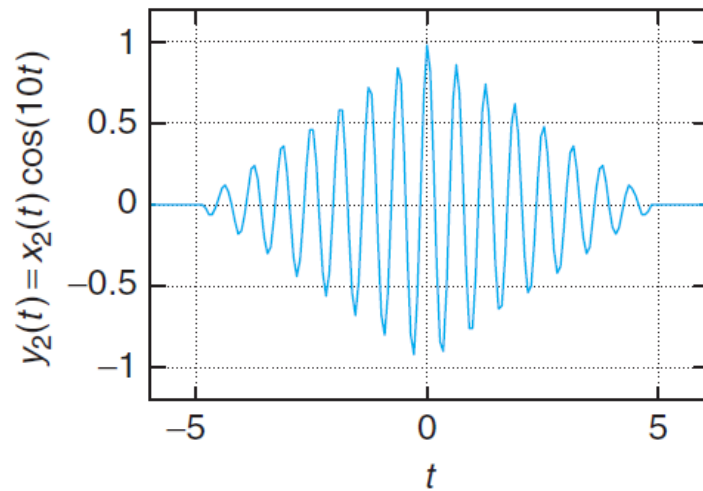
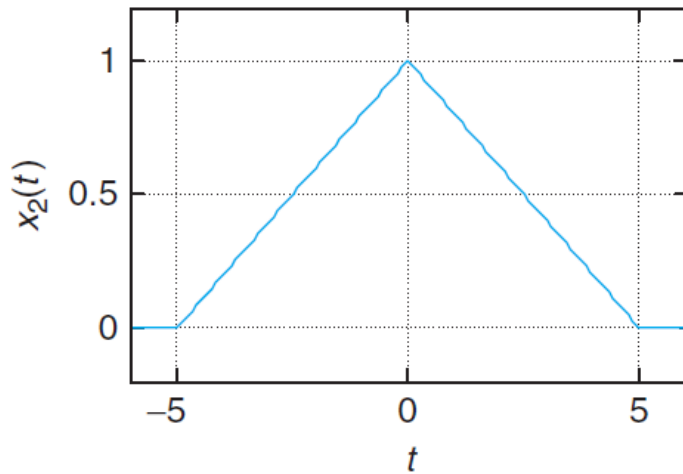
$$x(t) \cos(\Omega_0 t)$$

is given by

$$0.5 [X(\Omega - \Omega_0) + X(\Omega + \Omega_0)]$$

That is, $X(\Omega)$ is shifted to frequencies Ω_0 and $-\Omega_0$, and multiplied by 0.5.

Signal Modulation: Example



Fourier Transform of Periodic Signals

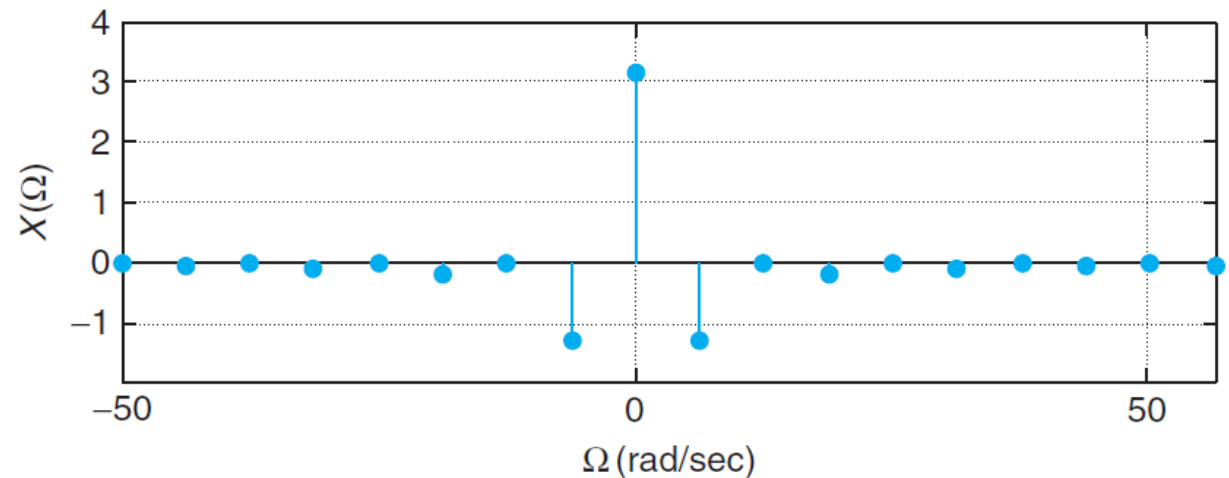
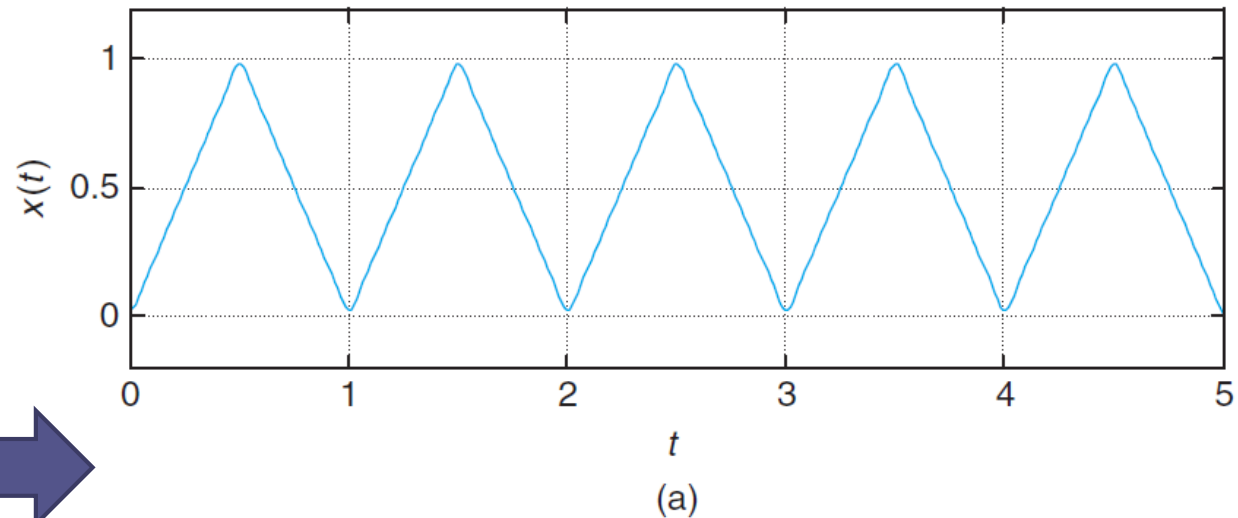
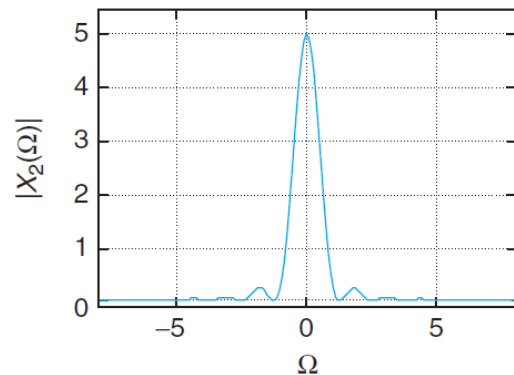
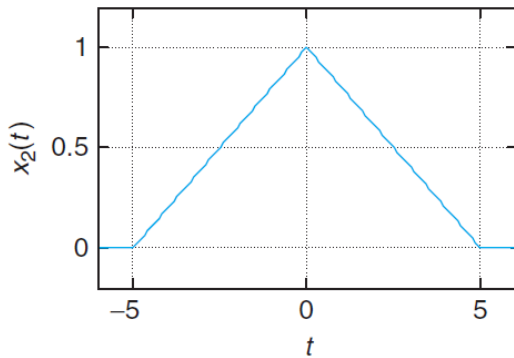
For a periodic signal $x(t)$ of period T_0 , we have the Fourier pair

$$x(t) = \sum_k X_k e^{jk\Omega_0 t} \quad \Leftrightarrow \quad X(\Omega) = \sum_k 2\pi X_k \delta(\Omega - k\Omega_0)$$

obtained by representing $x(t)$ by its Fourier series.

- Periodic Signals are represented by Sampled Fourier transform
- Sampled Signals are representing by Periodic Fourier Transform (from duality)

Fourier Transform of Periodic Signals: Example



Parseval's Energy Conservation

For a finite-energy signal $x(t)$ with Fourier transform $X(\Omega)$, its energy is conserved when going from the time to the frequency domain, or

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\Omega)|^2 d\Omega \quad (5.15)$$

Thus, $|X(\Omega)|^2$ is an energy density indicating the amount of energy at each of the frequencies Ω .

The plot $|X(\Omega)|^2$ versus Ω is called the energy spectrum of $x(t)$, and it displays how the energy of the signal is distributed over frequency.

- Energy in Time Domain = Energy in Frequency Domain

Symmetry of Spectral Representations

If $X(\Omega)$ is the Fourier transform of a real-valued signal $x(t)$, periodic or aperiodic, the magnitude $|X(\Omega)|$ is an even function of Ω :

$$|X(\Omega)| = |X(-\Omega)| \quad (5.16)$$

and the phase $\angle X(\Omega)$ is an odd function of Ω :

$$\angle X(\Omega) = -\angle X(-\Omega) \quad (5.17)$$

We then have:

Magnitude spectrum: $|X(\Omega)|$ versus Ω

Phase spectrum: $\angle X(\Omega)$ versus Ω

Energy/power spectrum: $|X(\Omega)|^2$ versus Ω

- Clearly, if the signal is complex, the above symmetry will **NOT** hold

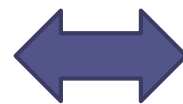
Convolution and Filtering

If the input $x(t)$ (periodic or aperiodic) to a stable LTI system has a Fourier transform $X(\Omega)$, and the system has a frequency response $H(j\Omega) = \mathcal{F}[h(t)]$ where $h(t)$ is the impulse response of the system, the output of the LTI system is the convolution integral $y(t) = (x * h)(t)$, with Fourier transform

$$Y(\Omega) = X(\Omega) H(j\Omega) \quad (5.18)$$

- Relation between transfer function and frequency response:

$$\begin{aligned} H(j\Omega) &= \mathcal{L}[h(t)]|_{s=j\Omega} \\ &= H(s)|_{s=j\Omega} \end{aligned}$$



$$H(j\Omega) = \frac{Y(\Omega)}{X(\Omega)}$$

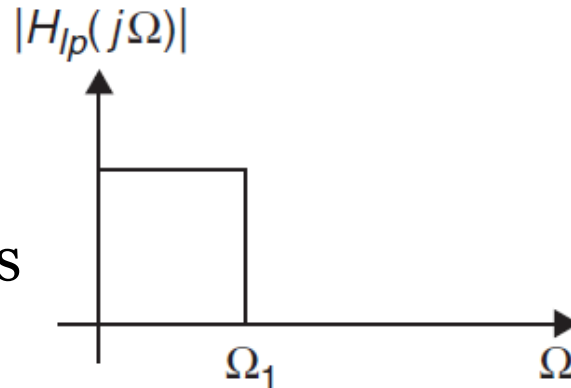
Basics of Filtering

- The filter design consists in finding a transfer function $H(s) = B(s)/A(s)$ that satisfies certain specifications that will allow getting rid of the noise. Such specifications are typically given in the frequency domain.

$$Y(\Omega) = H(j\Omega)X(\Omega)$$

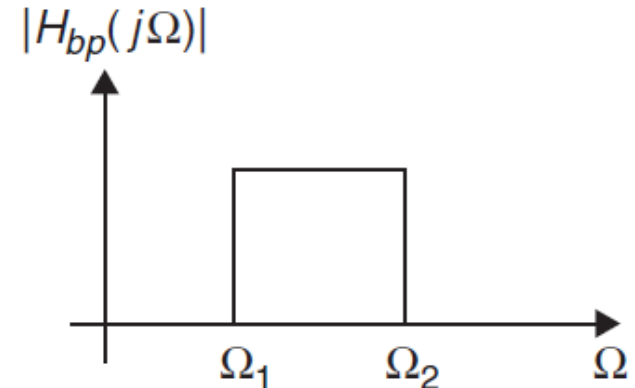
Ideal Filters

- (a) Low-Pass



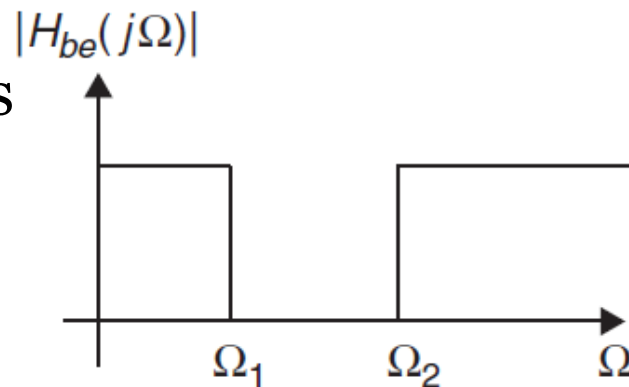
(a)

- (b) Band-Pass



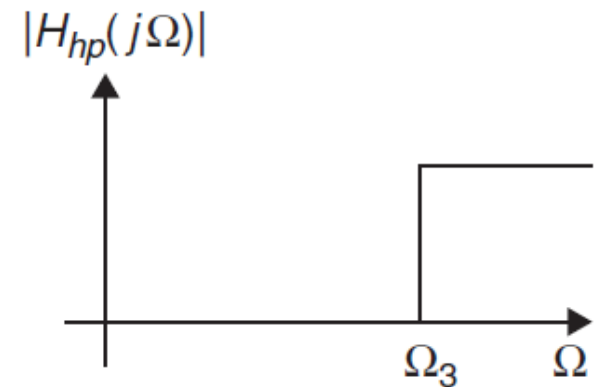
(b)

- (c) Band-Reject



(c)

- (d) High-Pass



(d)

Time Shifting Property

If $x(t)$ has a Fourier transform $X(\Omega)$, then

$$x(t - t_0) \Leftrightarrow X(\Omega)e^{-j\Omega t_0}$$

$$x(t + t_0) \Leftrightarrow X(\Omega)e^{j\Omega t_0}$$

- Example:

$$x(t) = A[\delta(t - \tau) + \delta(t + \tau)]$$



$$X(\Omega) = A[1e^{-j\Omega\tau} + 1e^{j\Omega\tau}]$$

Differentiation and Integration

If $x(t)$, $-\infty < t < \infty$, has a Fourier transform $X(\Omega)$, then

$$\frac{d^N x(t)}{dt^N} \Leftrightarrow (j\Omega)^N X(\Omega)$$
$$\int_{-\infty}^t x(\sigma) d\sigma \Leftrightarrow \frac{X(\Omega)}{j\Omega} + \pi X(0) \delta(\Omega)$$

where

$$X(0) = \int_{-\infty}^{\infty} x(t) dt$$

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Time shifting	$x(t - \alpha)$	$e^{-j\alpha\Omega} X(\Omega)$
Frequency shifting	$e^{j\Omega_0 t} x(t)$	$X(\Omega - \Omega_0)$
Modulation	$x(t) \cos(\Omega_c t)$	$0.5[X(\Omega - \Omega_c) + X(\Omega + \Omega_c)]$
Periodic signals	$x(t) = \sum_k X_k e^{jk\Omega_0 t}$	$X(\Omega) = \sum_k 2\pi X_k \delta(\Omega - k\Omega_0)$
Symmetry	$x(t) \text{ real}$	$ X(\Omega) = X(-\Omega) $ $\angle X(\Omega) = -\angle X(-\Omega)$
Convolution in time	$z(t) = [x * y](t)$	$Z(\Omega) = X(\Omega)Y(\Omega)$
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Covered Material and Assignments

- Chapter 5 of Chaparro's textbook
- Assigned Problems Set #4