EE 470 – Signals and Systems

8. Fourier Analysis of Discrete-Time Signals and Systems

Prof. Yasser Mostafa Kadah

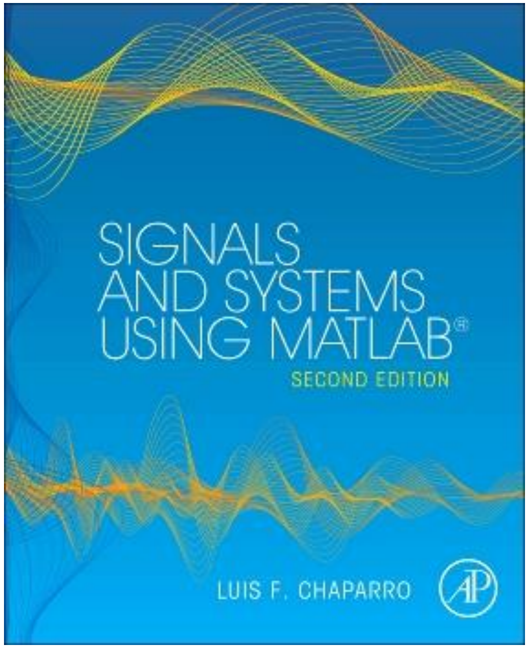
اقرأ باسم ربك

جامعة الملك عبدالعزيز

King Abdulaziz University

Textbook

Luis Chapparo, Signals and Systems Using Matlab, 2nd ed., Academic Press, 2015.



Discrete-Time Fourier Transform

- Sampled time domain signal has a Fourier transform that is periodic
- Periodic time domain signal has a Fourier transform that is sampled
- Periodic and sampled time domain signal has a Fourier transform that is both periodic and sampled

Forward DTFT

$$\boxed{X(e^{j\omega}) = \sum_n x[n]e^{-j\omega n} \qquad -\pi \le \omega < \pi}$$

Inverse DTFT

$$\boxed{x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n} d\omega}$$

Notes on DTFT

- DTFT is just one period of the continuous Fourier transform of the sampled signal
- Z-transform is related to the DTFT as follows:

$$\boxed{X_s(e^{j\omega}) = X(z)|_{z=e^{j\omega}}}$$

  - DTFT is the values on the unit circle in z-Transform
  - Unit circle must be included in the ROC
- Problem: the sampled signal is still infinite in length
  - Impossible to obtain using computers
  - Consider cases when the signal has compact support

Discrete Fourier Transform (DFT)

- Defined as the Fourier transform of a signal that is both discrete and periodic
  - Fourier transform will also be discrete and periodic
  - Can assume periodicity if we have a finite duration of a signal: better than zero assumption since it allows reducing the frequency domain to a sampled function

DFT Formula

- Given a periodic signal *x*[*n*] of period *N*, its DFT is given by,

$$\boxed{X[k] = \sum_{n=0}^{N-1} x[n] e^{-j2\pi nk/N} \qquad 0 \leq k \leq N-1}$$

- Its inverse is given by,

$$\boxed{x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j2\pi nk/N} \qquad 0 \leq n \leq N-1}$$

- Both *X*[*k*] and *x*[*n*] are periodic of the same period *N*.

Computation of DFT Using FFT

- A very efficient computation of the DFT is done by means of the FFT algorithm, which takes advantage of some special characteristics of the DFT
  - ▫ DFT requires $O(N^2)$ computations
  - ▫ FFT requires $O(N \log_2 N)$ computations
  - ▫ Both compute the same thing
- It should be understood that the FFT is not another transformation but an algorithm to efficiently compute DFTs.
- Always remember that the signal and its DFT are both periodic

Improving Resolution in DFT

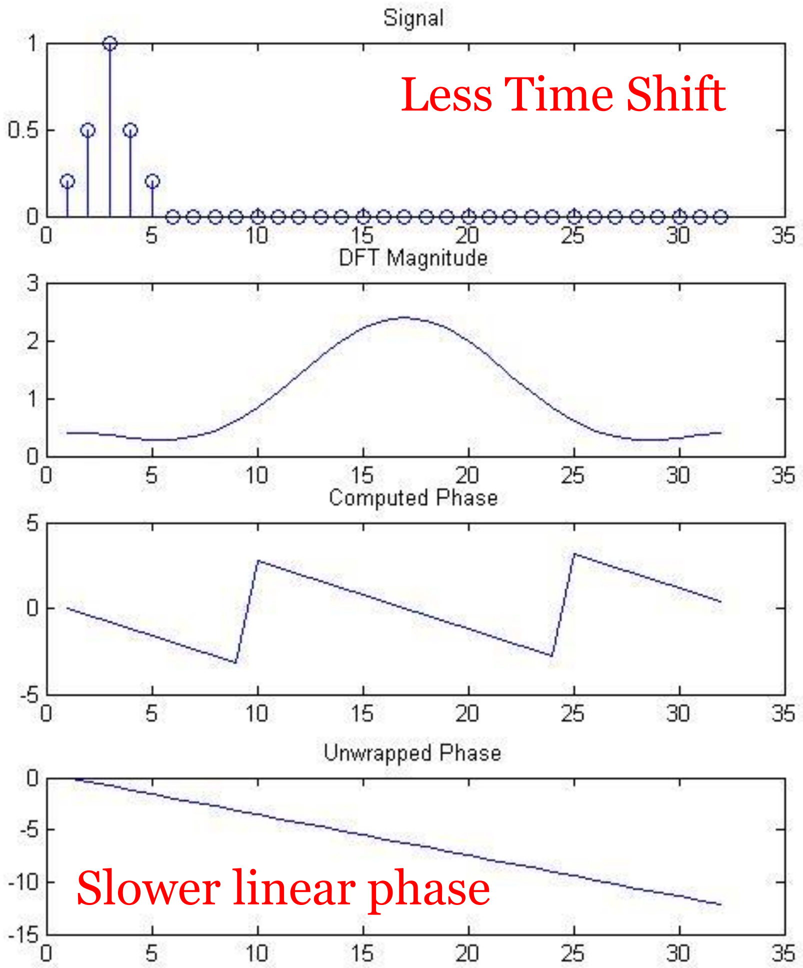
- When the signal $x[n]$ is periodic of period N, the DFT values are normalized Fourier series coefficients of $x[n]$ that only exist for the N harmonic frequencies $\{2\pi k/N\}$, as no frequency components exist for any other frequencies
  - ▫ The number of possible frequencies depend on the length L chosen to compute its DFT.
  - ▫ Frequencies at which we compute the DFT can be seen as frequencies around the unit circle in the z-plane
- We can improve the frequency resolution of its DFT by increasing the number of samples in the signal without distorting the signal by *padding the signal with zeros*
  - ▫ Increase samples of the units circle

Phase in DFT

- Output of DFT is complex numbers
- Phase is just as important as magnitude in DFT
- Time shift amounts to linear phase
  - Phase wrapping occurs for large shifts
- Linear phase can be unwrapped to its correct form using the “unwrap” function of Matlab

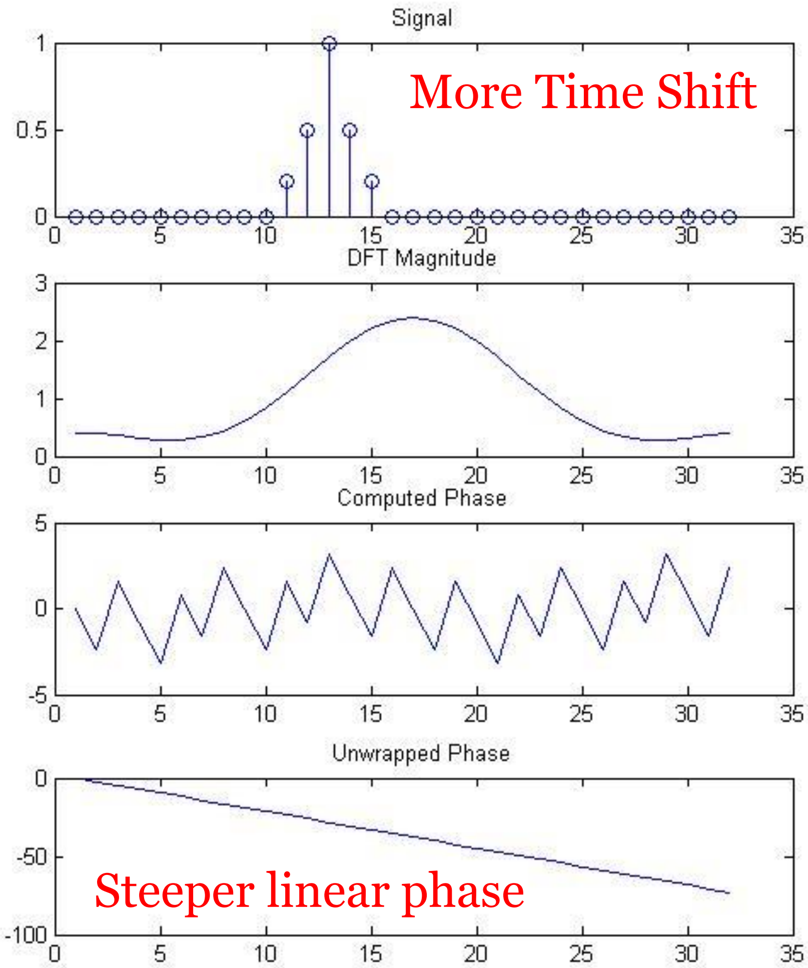
```
N= 32; x= zeros(N,1); x(11:15)= [0.2 0.5 1 0.5 0.2];
X= fftshift(fft(x));
subplot(4,1,1); stem(1:N,x);  title('Signal')
subplot(4,1,2); plot(abs(X));  title('DFT Magnitude')
subplot(4,1,3); plot(angle(X)); title('Computed Phase')
subplot(4,1,4); plot(unwrap(angle(X))); title('Unwrapped Phase')
```

Phase in DFT: Examples



Less Time Shift

Slower linear phase



More Time Shift

Steeper linear phase

Zero-Padding Example

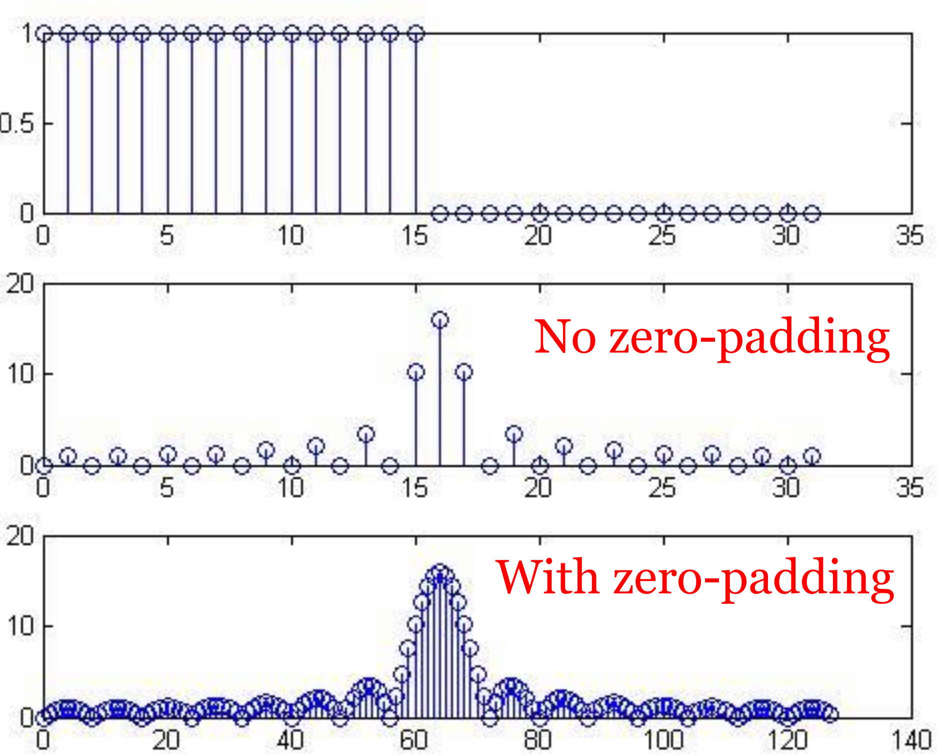
Matlab Code

```
N= 32;
N2= 128;
n=0:N-1;
n2= 0:N2-1;
x= zeros(N,1);
x(1:16)= 1;
X= fftshift(fft(x));
X2= fftshift(fft(x,N2));
subplot(3,1,1)
stem(n,x)
subplot(3,1,2)
stem(n,abs(X))
subplot(3,1,3)
stem(n2,abs(X2))
```

Original Signal

32-Point FFT

128-Point FFT



No zero-padding

With zero-padding

Linear vs. Circular Convolution

- Convolution in continuous and discrete time aperiodic signals is called *linear convolution*
- Convolution can be calculated as a multiplication in the frequency domain
  - Convolution property of the Fourier transform
- Problem: when using DFT to compute convolution of aperiodic signals, the periodicity assumption results in errors in computing the convolution
  - Signal outside the N points used is not zero: periodic extension of the N points disturb the computation
  - What we compute is called *circular convolution*

Linear vs. Circular Convolution

- Given $x[n]$ and $h[n]$ of lengths M and K, the linear convolution sum $y[n]$ of length $N=(M+K-1)$ can be found by following these three steps:
  - Compute DFTs $X[k]$ and $H[k]$ of length $L \geq N$ for $x[n]$ and $h[n]$ using zero-padding
  - Multiply them to get $Y[k]= X[k]H[k]$.
  - Find the inverse DFT of $Y[k]$ of length L to obtain $y[n]$

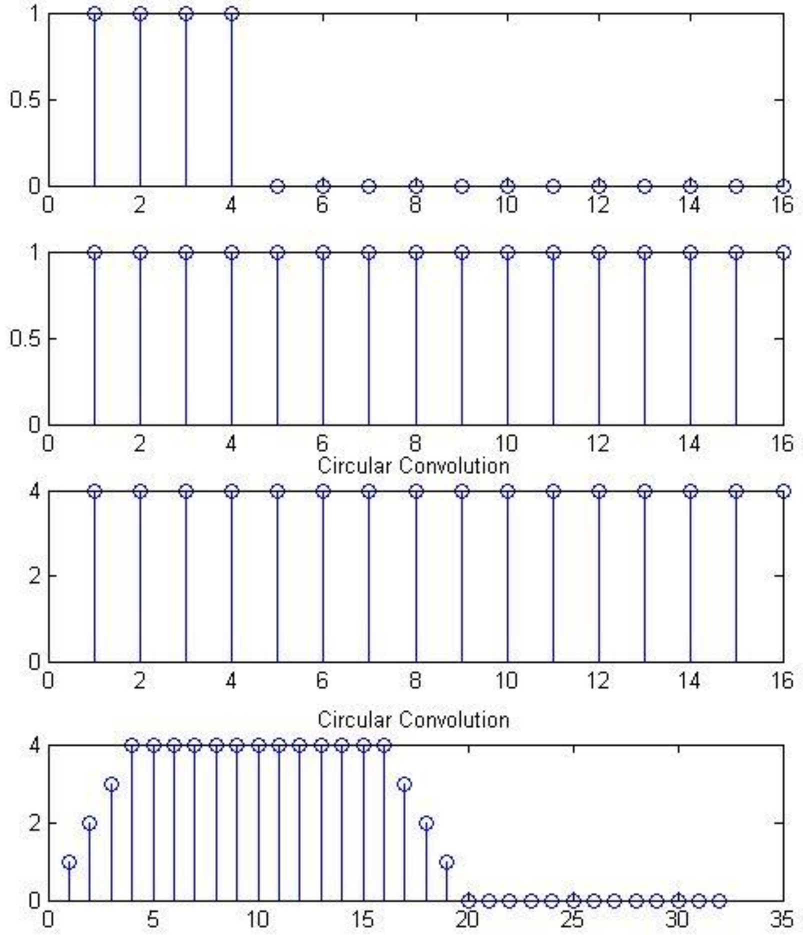
Linear vs. Circular Convolution: Example

```
N1= 16;
N2= 16;
N= 32;   % N>N1+N2-1
x1= zeros(N1,1);
x2= zeros(N2,1);
x1(1:4)= 1;
x2(:)= 1;

 X1= fft(x1,N1);
X2= fft(x2,N1);
X_circ= X1.*X2;
x_circ= ifft(X_circ);

X1= fft(x1,N);
X2= fft(x2,N);
X_lin= X1.*X2;
x_lin= ifft(X_lin);

figure(1); subplot(4,1,1); stem(1:N1,x1)
subplot(4,1,2); stem(1:N2,x2)
subplot(4,1,3); stem(1:N1, abs(x_circ))
title('Circular Convolution');
subplot(4,1,4); stem(1:N,abs(x_lin))
title('Circular Convolution');
```



Fourier Transformations Chart

Signal Classification		Frequency	
		Continuous	Discrete
Time	Continuous	Continuous Fourier Transform (CFT)	Discrete Fourier Series (DFS)
	Discrete	Discrete-Time Fourier Transform (DTFT)	Discrete Fourier Transform (DFT)

Most General Form:
others are special cases

Only one that can be done on a computer

Covered Material and Assignments

- Chapter 11 of Chaparro’s textbook
- Assigned Problem Set #8