EE 470 – Signals and Systems

3. The Laplace Transform

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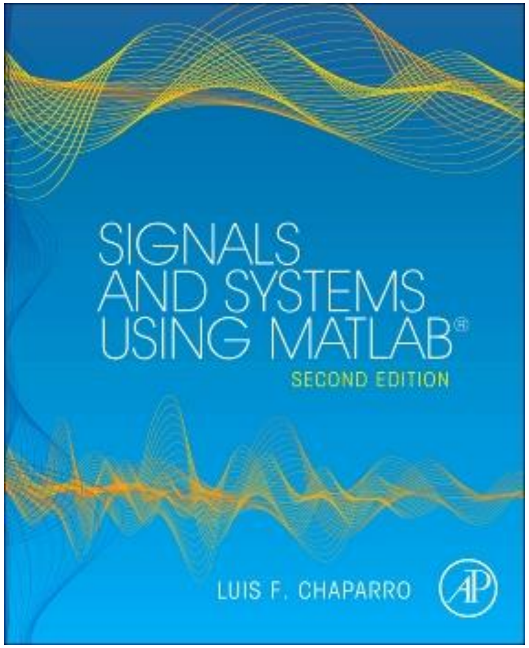
Textbook

Luis Chapparo, Signals and Systems Using Matlab, 2^{nd} ed., Academic Press, 2015.



Two-Sided Laplace Transform

The two-sided Laplace transform of a continuous-time function $f(t)$ is

$$F(s) = \mathcal{L}[f(t)] = \int_{-\infty}^{\infty} f(t)e^{-st}dt \qquad s \in \text{ROC} \tag{3.2}$$

where the variable $s = \sigma + j\Omega$, with Ω as the frequency in rad/sec and σ as a damping factor. ROC stands for the region of convergence—that is, where the integral exists.

The inverse Laplace transform is given by

$$f(t) = \mathcal{L}^{-1}[F(s)] = \frac{1}{2\pi j}\int_{\sigma - j\infty}^{\sigma + j\infty} F(s)e^{st}ds \qquad \sigma \in \text{ROC} \tag{3.3}$$

Two-Sided Laplace Transform

- Laplace transform F(s) provides a representation of f(t) in the s-domain, which in turn can be inverted back into the original time-domain function (1:1)
- Laplace transform of impulse response of an LTI system h(t) is H(s) and is called the system or *transfer function*
- Region of Convergence: region in s where transform exists

For the Laplace transform of $f(t)$ to exist we need that

$$\left|\int_{-\infty}^{\infty} f(t)e^{-st}dt\right| = \left|\int_{-\infty}^{\infty} f(t)e^{-\sigma t}e^{-j\Omega t}dt\right|$$

$$\leq \int_{-\infty}^{\infty} |f(t)e^{-\sigma t}|\, dt < \infty$$

or that $f(t)e^{-\sigma t}$ be absolutely integrable. This may be possible by choosing an appropriate σ even in the case when $f(t)$ is not absolutely integrable. The value chosen for σ determines the ROC of $F(s)$; the frequency Ω does not affect the ROC.

One-Sided Laplace Transform

The *one-sided Laplace transform* is defined as

$$F(s) = \mathcal{L}[f(t)u(t)] = \int_{0-}^{\infty} f(t)u(t)e^{-st}\,dt \tag{3.8}$$

where $f(t)$ is either a causal function or made into a causal function by the multiplication by $u(t)$. The one-sided Laplace transform is of significance given that most of the applications deal with causal systems and signals, and that any signal or system can be decomposed into causal and anti-causal components requiring only the computation of one-sided Laplace transforms.

Laplace Transformation Table

Table 3.1 One-Sided Laplace Transforms

	Function of Time	Function of s, ROC
1.	$\delta(t)$	1, whole s-plane
2.	$u(t)$	$\frac{1}{s}$, $\mathcal{R}e[s] > 0$
3.	$r(t)$	$\frac{1}{s^2}$, $\mathcal{R}e[s] > 0$
4.	$e^{-at}u(t),\ a > 0$	$\frac{1}{s+a}$, $\mathcal{R}e[s] > -a$
5.	$\cos(\Omega_0 t)u(t)$	$\frac{s}{s^2+\Omega_0^2}$, $\mathcal{R}e[s] > 0$
6.	$\sin(\Omega_0 t)u(t)$	$\frac{\Omega_0}{s^2+\Omega_0^2}$, $\mathcal{R}e[s] > 0$
7.	$e^{-at}\cos(\Omega_0 t)u(t),\ a > 0$	$\frac{s+a}{(s+a)^2+\Omega_0^2}$, $\mathcal{R}e[s] > -a$
8.	$e^{-at}\sin(\Omega_0 t)u(t),\ a > 0$	$\frac{\Omega_0}{(s+a)^2+\Omega_0^2}$, $\mathcal{R}e[s] > -a$
9.	$2A\,e^{-at}\cos(\Omega_0 t + \theta)u(t),\ a > 0$	$\frac{A\angle\theta}{s+a-j\Omega_0} + \frac{A\angle-\theta}{s+a+j\Omega_0}$, $\mathcal{R}e[s] > -a$
10.	$\frac{1}{(N-1)!}\,t^{N-1}u(t)$	$\frac{1}{s^N}$ N an integer, $\mathcal{R}e[s] > 0$
11.	$\frac{1}{(N-1)!}\,t^{N-1}e^{-at}u(t)$	$\frac{1}{(s+a)^N}$ N an integer, $\mathcal{R}e[s] > -a$
12.	$\frac{2A}{(N-1)!}\,t^{N-1}e^{-at}\cos(\Omega_0 t + \theta)u(t)$	$\frac{A\angle\theta}{(s+a-j\Omega_0)^N} + \frac{A\angle-\theta}{(s+a+j\Omega_0)^N}$, $\mathcal{R}e[s] > -a$

Laplace Transform Properties

Table 3.2 Basic Properties of One-Sided Laplace Transforms

Causal functions and constants	$\alpha f(t),\ \beta g(t)$	$\alpha F(s),\ \beta G(s)$
Linearity	$\alpha f(t) + \beta g(t)$	$\alpha F(s) + \beta G(s)$
Time shifting	$f(t-\alpha)$	$e^{-\alpha s}F(s)$
Frequency shifting	$e^{\alpha t} f(t)$	$F(s-\alpha)$
Multiplication by t	$t\, f(t)$	$-\frac{dF(s)}{ds}$
Derivative	$\frac{df(t)}{dt}$	$sF(s) - f(0-)$
Second derivative	$\frac{d^2 f(t)}{dt^2}$	$s^2F(s) - sf(0-) - f^{(1)}(0)$
Integral	$\int_{0-}^{t} f(t')dt$	$\frac{F(s)}{s}$
Expansion/contraction	$f(\alpha t)\ \alpha \neq 0$	$\frac{1}{\lvert\alpha\rvert} F\left(\frac{s}{\alpha}\right)$
Initial value	$f(0+) = \lim_{s\to\infty} sF(s)$	
Final value	$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$	

Linearity

For signals $f(t)$ and $g(t)$, with Laplace transforms $F(s)$ and $G(s)$, and constants a and b, we have the Laplace transform is linear:

$$\mathcal{L}[af(t)u(t) + bg(t)u(t)] = aF(s) + bG(s)$$

$$\mathcal{L}[af(t)u(t) + bg(t)u(t)] = \int_0^\infty [af(t) + bg(t)]u(t)e^{-st}dt$$

$$= a\int_0^\infty f(t)u(t)e^{-st}dt + b\int_0^\infty g(t)u(t)e^{-st}dt$$

$$= a\mathcal{L}[f(t)u(t)] + b\mathcal{L}[g(t)(t)]$$

Differentiation

For a signal $f(t)$ with Laplace transform $F(s)$ its one-sided Laplace transform of its first-and second-order derivatives are

$$\mathcal{L}\left[\frac{df(t)}{dt}u(t)\right] = sF(s) - f(0-) \tag{3.11}$$

$$\mathcal{L}\left[\frac{df(t)}{dt}u(t)\right] = \int_{0-}^{\infty} \frac{df(t)}{dt}e^{-st}dt$$

integral is evaluated by parts

$$\int w\,dv = wv - \int v\,dw$$

$$\int_{0-}^{\infty} \frac{df(t)}{dt}e^{-st}dt = e^{-st}f(t)\,\Big|_{0-}^{\infty} - \int_{0-}^{\infty} f(t)(-se^{-st})dt$$

$$= s\int_{0-}^{\infty} f(t)e^{-st}dt - f(0-)$$

$$= sF(s) - f(0-)$$

Integration

The Laplace transform of the integral of a causal signal $y(t)$ is given by

$$\mathcal{L}\left[\int_0^t y(\tau)d\tau \;\; u(t)\right] = \frac{Y(s)}{s} \qquad (3.14)$$

$$f(t) = \int_0^t y(\tau)d\tau u(t) \;\Rightarrow\; \frac{df(t)}{dt} = y(t)u(t)$$

$$\mathcal{L}\left[\frac{df(t)}{dt}\right] = sF(s) - f(0) = Y(s) \;\Rightarrow\; F(s) = \mathcal{L}\left[\int_0^t y(\tau)d\tau\right] = \frac{Y(s)}{s}$$

Time Shifting

If the Laplace transform of $f(t)u(t)$ is $F(s)$, the Laplace transform of the time-shifted signal $f(t-\tau)u(t-\tau)$ is

$$\mathcal{L}[f(t-\tau)u(t-\tau)] = e^{-\tau s}F(s) \tag{3.15}$$

- Simply shown by a change of variables

Convolution Integral

The Laplace transform of the convolution integral of a causal signal $x(t)$, with Laplace transforms $X(s)$, and a causal impulse response $h(t)$, with Laplace transform $H(s)$, is given by

$$\mathcal{L}[(x * h)(t)] = X(s)H(s) \tag{3.16}$$

$$y(t) = \int_0^\infty x(\tau)h(t-\tau)d\tau \qquad t \geq 0$$

$$Y(s) = \mathcal{L}\left[\int_0^\infty x(\tau)h(t-\tau)d\tau\right] = \int_0^\infty \left[\int_0^\infty x(\tau)h(t-\tau)d\tau\right] e^{-st}dt$$

$$= \int_0^\infty x(\tau)\left[\int_0^\infty h(t-\tau)\; e^{-s(t-\tau)}\; dt\right] e^{-s\tau}d\tau = X(s)H(s)$$

$$\Rightarrow \boxed{H(s) = \frac{\mathcal{L}[y(t)]}{\mathcal{L}[x(t)]} = \frac{\mathcal{L}[\text{ output }]}{\mathcal{L}[\text{ input }]}}$$

Example 1

Find the Laplace transforms of $\delta(t)$, $u(t)$, and a pulse $p(t) = u(t) - u(t-1)$.

$$\mathcal{L}[\delta(t)] = \int_{-\infty}^{\infty} \delta(t)e^{-st}dt = \int_{-\infty}^{\infty} \delta(t)e^{-s0}dt = \int_{-\infty}^{\infty} \delta(t)dt = 1$$

$$U(s) = \mathcal{L}[u(t)] = \int_{-\infty}^{\infty} u(t)e^{-st}dt = \int_{0}^{\infty} e^{-st}dt = \int_{0}^{\infty} e^{-\sigma t}e^{-j\Omega t}dt \Rightarrow U(s) = \frac{e^{-st}}{-s}\Big|_{t=0}^{\infty} = \frac{1}{s}$$

$$P(s) = \mathcal{L}[u(t+1) - u(t-1)] = \int_{-1}^{1} e^{-st}dt = \frac{-e^{-st}}{s}\Big|_{t=-1}^{1} = \frac{1}{s}[e^{s} - e^{-s}] = \frac{e^{s}}{s}[1 - e^{-2s}]$$

Example 2

- Compute H(s) for:

$$h(t) = e^{-t}u(t) + e^{2t}u(-t)$$
$$= h_c(t) + h_{ac}(t)$$

- Using table:
  - causal component: $$H_c(s) = \frac{1}{s+1} \qquad \sigma > -1$$
  - Anti-causal component: $$\mathcal{L}[h_{ac}(t)] = \mathcal{L}[h_{ac}(-t)u(t)]_{(-s)} = \frac{1}{-s+2}$$
    $$\sigma < 2$$

$$\Downarrow$$

$$H(s) = \frac{1}{s+1} + \frac{1}{-s+2} = \frac{-3}{(s+1)(s-2)} \qquad -1 < \sigma < 2$$

Example 3

Compute the Laplace transform of the ramp function $r(t) = tu(t)$ and use it to find the Laplace of a triangular pulse $\Lambda(t) = r(t+1) - 2r(t) + r(t-1)$.

$$R(s) = \int_0^{\infty} te^{-st}dt = \frac{e^{-st}}{s^2}(-st-1)\Big|_{t=0}^{\infty} = \frac{1}{s^2} \qquad \sigma > 0$$

$$\Rightarrow \qquad \Lambda(s) = \frac{1}{s^2}[e^s - 2 + e^{-s}] \qquad \sigma > 0$$

Example 4

- Use the differentiation property to compute the Laplace transformation of $\delta(t)$, $u(t)$, and $r(t)$ starting from R(s) derived in Example 3

$$\mathcal{L}[r(t)] = \frac{1}{s^2}$$

$$\mathcal{L}\left[u(t) = \frac{dr(t)}{dt}\right] = s\frac{1}{s^2} = \frac{1}{s}$$

$$\mathcal{L}\left[\delta(t) = \frac{du(t)}{dt}\right] = s\frac{1}{s} = 1$$

Example 5

- Let y(t) be a causal signal. Compute Y(s) given that

$$\int_0^t y(\tau)d\tau = 3u(t) - 2y(t)$$

$$\Downarrow$$

$$\frac{Y(s)}{s} = \frac{3}{s} - 2Y(s) \quad \Rightarrow \quad Y(s) = \frac{3}{2(s+0.5)}$$

Laplace Transform ROC

The Laplace transform of a

- Finite support function (i.e., $f(t) = 0$ for $t < t_1$ and $t > t_2$, for $t_1 < t_2$) is

$$\mathcal{L}[f(t)] = \mathcal{L}\left[f(t)[u(t-t_1) - u(t-t_2)]\right] \qquad \text{whole } s\text{-plane}$$

- Causal function (i.e., $f(t) = 0$ for $t < 0$) is

$$\mathcal{L}[f(t)u(t)] \qquad \mathcal{R}_c = \{(\sigma, \Omega) : \sigma > \max\{\sigma_i\}, -\infty < \Omega < \infty\}$$

- Anti-causal function (i.e., $f(t) = 0$ for $t > 0$) is

$$\mathcal{L}[f(t)u(-t)] \qquad \mathcal{R}_{ac} = \{(\sigma, \Omega) : \sigma < \min\{\sigma_i\}, -\infty < \Omega < \infty\}$$

- Noncausal function (i.e., $f(t) = f_{ac}(t) + f_c(t) = f(t)u(-t) + f(t)u(t)$) is

$$\mathcal{L}[f(t)] = \mathcal{L}[f_{ac}(-t)u(t)]_{(-s)} + \mathcal{L}[f_c(t)u(t)] \qquad \mathcal{R}_c \bigcap \mathcal{R}_{ac}$$

Inverse Laplace Transform

- Inverting the Laplace transform consists in finding a signal that has the given transform with the given region of convergence (ROC)
- 3 Cases:
  - Inverse of one-sided Laplace transforms giving causal functions
  - Inverse of Laplace transforms with exponentials.
  - Inverse of two-sided Laplace transforms giving anti-causal or noncausal functions

Inverse of One-Sided Laplace Transforms: Partial Fraction Expansion

- Expanding the given function in s into a sum of components of which the inverse Laplace transforms can be found in a table of Laplace transform pairs

$$\boxed{X(s) = \frac{N(s)}{D(s)}} \Rightarrow X(s) = g_0 + g_1 s + \cdots + g_m s^m + \frac{B(s)}{D(s)}$$

$$\Big\Downarrow \mathcal{L}^{-1}$$

$$x(t) = g_0\delta(t) + g_1\frac{d\delta(t)}{dt} + \cdots + g_m\frac{d^m\delta(t)}{dt^m} + \mathcal{L}^{-1}\left[\frac{B(s)}{D(s)}\right]$$

Simple Real Poles

$$X(s) = \frac{N(s)}{D(s)} = \frac{N(s)}{\prod_k (s - p_k)}$$

$$X(s) = \sum_k \frac{A_k}{s - p_k}$$

$$A_k = X(s)(s - p_k)\Big|_{s=p_k}$$

$\mathcal{L}^{-1}$

$$\boxed{x(t) = \sum_k A_k e^{p_k t} u(t)}$$

Table 3.1 One-Sided Laplace Transforms

	Function of Time	Function of s, ROC
1.	$\delta(t)$	1, whole s-plane
2.	$u(t)$	$\frac{1}{s}$, $\mathcal{Re}[s] > 0$
3.	$r(t)$	$\frac{1}{s^2}$, $\mathcal{Re}[s] > 0$
4.	$e^{-at}u(t),\ a > 0$	$\frac{1}{s+a}$, $\mathcal{Re}[s] > -a$
5.	$\cos(\Omega_0 t)u(t)$	$\frac{s}{s^2+\Omega_0^2}$, $\mathcal{Re}[s] > 0$
6.	$\sin(\Omega_0 t)u(t)$	$\frac{\Omega_0}{s^2+\Omega_0^2}$, $\mathcal{Re}[s] > 0$
7.	$e^{-at}\cos(\Omega_0 t)u(t),\ a > 0$	$\frac{s+a}{(s+a)^2+\Omega_0^2}$, $\mathcal{Re}[s] > -a$
8.	$e^{-at}\sin(\Omega_0 t)u(t),\ a > 0$	$\frac{\Omega_0}{(s+a)^2+\Omega_0^2}$, $\mathcal{Re}[s] > -a$
9.	$2A\, e^{-at}\cos(\Omega_0 t + \theta)u(t),\ a > 0$	$\frac{A\angle\theta}{s+a-j\Omega_0} + \frac{A\angle-\theta}{s+a+j\Omega_0}$, $\mathcal{Re}[s] > -a$
10.	$\frac{1}{(N-1)!}\ t^{N-1}u(t)$	$\frac{1}{s^N}$ N an integer, $\mathcal{Re}[s] > 0$
11.	$\frac{1}{(N-1)!}\ t^{N-1}e^{-at}u(t)$	$\frac{1}{(s+a)^N}$ N an integer, $\mathcal{Re}[s] > -a$
12.	$\frac{2A}{(N-1)!}\ t^{N-1}e^{-at}\cos(\Omega_0 t + \theta)u(t)$	$\frac{A\angle\theta}{(s+a-j\Omega_0)^N} + \frac{A\angle-\theta}{(s+a+j\Omega_0)^N}$, $\mathcal{Re}[s] > -a$

Simple Real Poles – Example

- Find the causal inverse of:

$$X(s) = \frac{3s+5}{s^2+3s+2} = \frac{3s+5}{(s+1)(s+2)}$$

$$X(s) = \frac{A_1}{s+1} + \frac{A_2}{s+2} \xrightarrow{\mathcal{L}^{-1}} x(t) = [A_1 e^{-t} + A_2 e^{-t}]u(t)$$

$$= [2e^{-t} + e^{-2t}]u(t)$$

$$A_1 = X(s)(s+1)|_{s=-1} = \frac{3s+5}{s+2}|_{s=-1} = 2$$

$$A_2 = X(s)(s+2)|_{s=-2} = \frac{3s+5}{s+1}|_{s=-2} = 1$$

Simple Complex Conjugate Poles

$$\boxed{X(s) = \frac{N(s)}{(s+\alpha)^2 + \Omega_0^2} = \frac{N(s)}{(s+\alpha-j\Omega_0)(s+\alpha+j\Omega_0)}}$$

$$\Downarrow$$

$$X(s) = \frac{A}{s+\alpha-j\Omega_0} + \frac{A^*}{s+\alpha+j\Omega_0}$$

$$A = X(s)(s+\alpha-j\Omega_0)|_{s=-\alpha+j\Omega_0} = |A|e^{j\theta}$$

$$\Downarrow \mathcal{L}^{-1}$$

$$\boxed{x(t) = 2|A|e^{-\alpha t}\cos(\Omega_0 t + \theta)u(t)}$$

Simple Complex Conjugate Poles

- Fine the causal inverse Laplace transform of

$$X(s) = \frac{2s+3}{s^2+2s+4} = \frac{2s+3}{(s+1)^2+3}$$

Table 3.1 One-Sided Laplace Transforms

	Function of Time	Function of s, ROC
1.	$\delta(t)$	1, whole s-plane
2.	$u(t)$	$\frac{1}{s}$, $\mathcal{R}e[s] > 0$
3.	$r(t)$	$\frac{1}{s^2}$, $\mathcal{R}e[s] > 0$
4.	$e^{-at}u(t),\ a > 0$	$\frac{1}{s+a}$, $\mathcal{R}e[s] > -a$
5.	$\cos(\Omega_0 t)u(t)$	$\frac{s}{s^2+\Omega_0^2}$, $\mathcal{R}e[s] > 0$
6.	$\sin(\Omega_0 t)u(t)$	$\frac{\Omega_0}{s^2+\Omega_0^2}$, $\mathcal{R}e[s] > 0$
7.	$e^{-at}\cos(\Omega_0 t)u(t),\ a > 0$	$\frac{s+a}{(s+a)^2+\Omega_0^2}$, $\mathcal{R}e[s] > -a$
8.	$e^{-at}\sin(\Omega_0 t)u(t),\ a > 0$	$\frac{\Omega_0}{(s+a)^2+\Omega_0^2}$, $\mathcal{R}e[s] > -a$
9.	$2A\, e^{-at}\cos(\Omega_0 t + \theta)u(t),\ a > 0$	$\frac{A\angle\theta}{s+a-j\Omega_0} + \frac{A\angle-\theta}{s+a+j\Omega_0}$, $\mathcal{R}e[s] > -a$
10.	$\frac{1}{(N-1)!}\, t^{N-1}u(t)$	$\frac{1}{s^N}$ N an integer, $\mathcal{R}e[s] > 0$
11.	$\frac{1}{(N-1)!}\, t^{N-1}e^{-at}u(t)$	$\frac{1}{(s+a)^N}$ N an integer, $\mathcal{R}e[s] > -a$
12.	$\frac{2A}{(N-1)!}\, t^{N-1}e^{-at}\cos(\Omega_0 t + \theta)u(t)$	$\frac{A\angle\theta}{(s+a-j\Omega_0)^N} + \frac{A\angle-\theta}{(s+a+j\Omega_0)^N}$, $\mathcal{R}e[s] > -a$

$$X(s) = \frac{1}{\sqrt{3}}\frac{\sqrt{3}}{(s+1)^2+3} + 2\frac{s+1}{(s+1)^2+3}$$

$$\Downarrow \mathcal{L}^{-1}$$

$$\boxed{x(t) = \left[\frac{1}{\sqrt{3}}\sin(\sqrt{3}t) + 2\cos(\sqrt{3}t)\right]e^{-t}u(t)}$$

Double Real Poles

$$X(s) = \frac{N(s)}{(s+\alpha)^2} = \frac{a + b(s+\alpha)}{(s+\alpha)^2} = \frac{a}{(s+\alpha)^2} + \frac{b}{s+\alpha}$$

$$a = X(s)(s+\alpha)^2 \,|_{s=-\alpha}$$

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5.	$\cos(\Omega_0 t)u(t)$	$\frac{s}{s^2+\Omega_0^2}$, $\mathcal{R}e[s] > 0$
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7.	$e^{-at}\cos(\Omega_0 t)u(t),\ a > 0$	$\frac{s+a}{(s+a)^2+\Omega_0^2}$, $\mathcal{R}e[s] > -a$
8.	$e^{-at}\sin(\Omega_0 t)u(t),\ a > 0$	$\frac{\Omega_0}{(s+a)^2+\Omega_0^2}$, $\mathcal{R}e[s] > -a$
9.	$2A\, e^{-at}\cos(\Omega_0 t + \theta)u(t),\ a > 0$	$\frac{A\angle\theta}{s+a-j\Omega_0} + \frac{A\angle-\theta}{s+a+j\Omega_0}$, $\mathcal{R}e[s] > -a$
10.	$\frac{1}{(N-1)!}\, t^{N-1}u(t)$	$\frac{1}{s^N}$ N an integer, $\mathcal{R}e[s] > 0$
11.	$\frac{1}{(N-1)!}\, t^{N-1}e^{-at}u(t)$	$\frac{1}{(s+a)^N}$ N an integer, $\mathcal{R}e[s] > -a$
12.	$\frac{2A}{(N-1)!}\, t^{N-1}e^{-at}\cos(\Omega_0 t + \theta)u(t)$	$\frac{A\angle\theta}{(s+a-j\Omega_0)^N} + \frac{A\angle-\theta}{(s+a+j\Omega_0)^N}$, $\mathcal{R}e[s] > -a$

$\mathcal{L}^{-1}$

$$\boxed{x(t) = [ate^{-\alpha t} + be^{-\alpha t}]u(t)}$$

Double Real Poles - Example

- Find the causal inverse Laplace transform of:

$$X(s) = \frac{4}{s(s+2)^2}$$

$$X(s) = \frac{A}{s} + \frac{a + b(s+2)}{(s+2)^2} \Rightarrow \boxed{x(t) = [1 - 2te^{-2t} - e^{-2t}]u(t)}$$

$$A = X(s)s|_{s=0} = 1$$

$$X(s) - \frac{1}{s} = \frac{4 - (s+2)^2}{s(s+2)^2} = \frac{-(s+4)}{(s+2)^2}$$

$$= \frac{a + b(s+2)}{(s+2)^2}$$

$$\Rightarrow \quad b = -1 \qquad a = -2$$

Analysis of LTI Systems

- Complete response for system described by

$$y^{(N)}(t) + \sum_{k=0}^{N-1} a_k y^{(k)}(t) = \sum_{\ell=0}^{M} b_\ell x^{(\ell)}(t) \qquad N > M$$

with initial conditions: $\{y^{(k)}(t),\ 0 \le k \le N-1\}$

is obtained by inverting the Laplace transform:

$$\boxed{Y(s) = \frac{B(s)}{A(s)} X(s) + \frac{1}{A(s)} I(s)}$$

$$A(s) = \sum_{k=0}^{N} a_k s^k \qquad a_N = 1$$

$$B(s) = \sum_{\ell=0}^{M} b_\ell s^\ell$$

$$I(s) = \sum_{k=1}^{N} a_k \left(\sum_{m=0}^{k-1} s^{k-m-1} y^{(m)}(0) \right)$$

Analysis of LTI Systems

- Let $H(s) = \frac{B(s)}{A(s)}$ and $H_1(s) = \frac{1}{A(s)}$

$$Y(s) = H(s)X(s) + H_1(s)I(s) \Rightarrow y(t) = y_{zs}(t) + y_{zi}(t)$$

$$\boxed{\begin{array}{ll} \textit{zero-state response:} & y_{zs}(t) = \mathcal{L}^{-1}[H(s)X(s)] \\ \textit{zero-input response:} & y_{zi}(t) = \mathcal{L}^{-1}[H_1(s)I(s)] \end{array}}$$

Analysis of LTI Systems - Example

- Find the impulse response h(t) and unit step response s(t) of the system defined by the differential equation:

$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = x(t)$$

- Assume zero initial conditions (LTI)

$$Y(s)[s^2 + 3s + 2] = X(s)$$

$$H(s) = \frac{1}{s^2 + 3s + 2} = \frac{1}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2} \qquad S(s) = \frac{H(s)}{s}$$

$$A = 1 \text{ and } B = -1$$

$$\boxed{h(t) = \left[e^{-t} - e^{-2t}\right] u(t)}$$

$$\boxed{s(t) = 0.5u(t) - e^{-t}u(t) + 0.5e^{-2t}u(t)}$$

Analysis of LTI Systems - Example

- Consider the same system but with nonzero initial conditions

$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = x(t) \qquad y(0) = 1 \text{ and } dy(t)/dt|_{t=0} = 0$$

- Can we compute h(t) ??

$$[s^2Y(s) - sy(0) - \frac{dy(t)}{dt}|_{t=0}] + 3[sY(s) - y(0)] + 2Y(s) = X(s)$$

$$Y(s)(s^2 + 3s + 2) - (s + 3) = X(s)$$

Unable to find H(s)=Y(s)/X(s)!

$$Y(s) = \frac{X(s)}{(s+1)(s+2)} + \frac{s+3}{(s+1)(s+2)}$$

Computation of the Convolution Integral

The Laplace transform of the convolution $y(t) = [x * h](t)$ is given by the product

$$Y(s) = X(s)H(s)$$

where $X(s) = \mathcal{L}[x(t)]$ and $H(s) = \mathcal{L}[h(t)]$. The transfer function *of the system* $H(s)$ is defined as

$$H(s) = \mathcal{L}[h(t)] = \frac{Y(s)}{X(s)}$$

- $H(s)$ transfers the Laplace transform $X(s)$ of the input into the Laplace transform of the output $Y(s)$.
- Once $Y(s)$ is found, $y(t)$ is computed by means of the inverse Laplace transform.

Computation of the Convolution Integral - Example

Use the Laplace transform to find the convolution $y(t) = [x * h](t)$ when

(1) the input is $x(t) = u(t)$ and the impulse response is a pulse $h(t) = u(t) - u(t-1)$, and
(2) the input and the impulse response of the system are $x(t) = h(t) = u(t) - u(t-1)$.

The Laplace transforms are $X(s) = \mathcal{L}[u(t)] = 1/s$ and $H(s) = \mathcal{L}[h(t)] = (1 - e^{-s})/s$, so that

$$Y(s) = H(s)X(s) = \frac{1 - e^{-s}}{s^2} \xrightarrow{\mathcal{L}^{-1}} y(t) = r(t) - r(t-1)$$

In the second case, $X(s) = H(s) = \mathcal{L}[u(t) - u(t-1)] = (1 - e^{-s})/s$, so that

$$Y(s) = H(s)X(s) = \frac{(1 - e^{-s})^2}{s^2} = \frac{1 - 2e^{-s} + e^{-2s}}{s^2} \xrightarrow{\mathcal{L}^{-1}} y(t) = r(t) - 2r(t-1) + r(t-2)$$

Covered Material and Assignments

- Chapter 3 of Chaparro's textbook
- Assigned Problem Set #3