New solution to the gridding problem

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ABSTRACT

Image reconstruction from nonuniformly sampled frequency domain data is an important problem that arises in computed imaging. The current reconstruction techniques suffer from fundamental limitations in their model and implementation that result in blurred reconstruction and/or artifacts. Here, we present a new approach for solving this problem that relies on a more realistic model and involves an explicit measure for the reconstruction accuracy that is optimized iteratively. The image is assumed piecewise constant to impose practical display constraints using pixels. We express the mapping of these unknown pixel values to the available frequency domain values as a linear system. Even though the system matrix is shown to be dense and too large to solve for practical purposes, we observe that applying a simple orthogonal transformation to the rows of this matrix converts the matrix into a sparse format. The transformed system is subsequently solved using the conjugate gradient method. The proposed method is applied to reconstruct images of a numerical phantom as well as actual magnetic resonance images using spiral sampling. The results support the theory and show that the computational load of this method is similar to that of other techniques. This suggests its potential for practical use.

Keywords: Gridding, image reconstruction, sparse linear systems, conjugate gradient method.

1. INTRODUCTION

The use of tomographic reconstruction has become an important tool for all advanced imaging modalities such as magnetic resonance imaging (MRI) and computerized tomography (CT). The use of these modalities offers accurate means for noninvasive diagnosis in medicine by providing images of exquisite accuracy and detail. This is the reason why the use of such modalities is fast expanding. In particular, the use of magnetic resonance imaging in medicine has extended to applications that have never been attempted before using other imaging modalities. Given its unique dependence of its signal on both the physical and chemical structures of the body as well as the physiological function, this technique holds the most potential among the available imaging techniques today.

Among the important goals of tomographic modalities such as magnetic resonance imaging is to become capable of performing accurate real-time imaging. This imposes a demanding constraint on the processes of image acquisition and reconstruction. As a result, a considerable amount of research effort has been directed towards making these processes faster and more efficient without sacrificing the quality of the image. We will consider here such efforts in the area of magnetic resonance imaging without loss of generality to other tomographic modalities.

In magnetic resonance imaging, the acquisition of images using fast acquisition techniques such as EPI is limited in speed by how fast the switching of gradients can be achieved. As a result, in order to speed up the acquisition process, nonrectilinear k-space trajectories such as spiral trajectory have been introduced to enable smoother gradient waveforms. Even though this lead to a more efficient acquisition process besides the reduction of inhomogeneity and motion artifacts, the process of reconstructing images from the nonuniformly sampled k-space associated with such sequences remained less than optimal in speed and accuracy. Given that conventional MRI relies on 2-D discrete Fourier transformation as its reconstruction tool, all attempts to address the problem of estimating a uniformly sampled k-space given the original nonuniformly sampled version. This problem is conventionally called the *gridding* problem. In his seminal work published in 1985, O'Sullivan concluded that the ideal resampling kernel is theoretically in the form of an infinite Sinc function and that the convolution with such a kernel followed by sampling on a

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Medical Imaging 2002: Image Processing, Milan Sonka, J. Michael Fitzpatrick,
Editors, Proceedings of SPIE Vol. 4684 (2002) © 2002 SPIE · 1605-7422/02/$15.00

rectilinear grid results in accurate reconstruction. Furthermore, the optimization of an error-minimizing objective function based on a truncated kernel results in the selection of prolate-spheroidal functions as the optimal kernel functions for such process. Such functions can be well approximated in the discrete domain by the Kaiser-Bessel window. This procedure has been utilized in practice as the classical gridding technique for the past decade. Several research papers addressed several aspects of this original method including the optimal selection of the width and form of the gridding kernel for spiral imaging[3], the practical implementation algorithm and applications to clinical studies[4,5], the compensation of sampling density variations[6], the efficient use of computer arithmetic for faster computation[7], and the optimized selection of sampling density compensation factors[8]. The performance of nonrectilinear algorithms in the presence of inhomogeneities was also shown to be more robust[9]. Therefore, it is fair to say that the use of such gridding method is well established among the MRI community. Nevertheless, several issues remained as open questions in using such method that include the choice of kernel size and shape for different k-space trajectories, the possible consequences of undersampling some areas in the k-space, and the reduction of some of the characteristic reconstruction artifacts that became apparent with gridding. This led to the introduction of another approach to this problem that attempts to achieve optimality in some sense for the reconstruction process.

The new approaches to solving the gridding problem pose the problem of estimating the rectilinear points from the nonrectilinear ones as a solution to a linear system of equations. This system of equations has a rather large size that is equal to the total number of points. That is, to obtain a gridded N×N k-space, a linear system of size N^2 must be solved. Even though the solution to this problem will in fact be the optimal solution to the gridding problem, it is not feasible to implement in practice for both storage size and computation time reasons. As a result, a number of attempts have been directed towards obtaining approximate solutions to this system of equations with realizable implementation requirements. Examples of such methods include the decomposition of the system matrix into local small matrix parts that can be solved using SVD[10], evaluation of such method and its variants in gridding different trajectories[11,12], and the transformation of the system matrix into a structured matrix form for faster computation[13]. Moreover, some authors attempted to estimate the continuous image from the sampled k-space using similar techniques[14]. Even though such methods offer as advantage over the conventional gridding method, they do not represent an optimal solution to the original matrix equation. In particular, there is no direct way of gauging how accurate the solution is within the formulation of such techniques. Moreover, the reconstruction in such methods is rigid in the sense that the relation between computation time and the accuracy is fixed. One cannot trade off accuracy and computation time to customize the use of such methods in different applications that might be sensitive to one of them more than the other. This is particularly important given the general consensus that gridding techniques are perceived as slow. Therefore, another solution strategy that offers to overcome such limitations would certainly be advantageous.

In this work, we describe a novel solution to the gridding problem based on a more realistic model of the problem. Instead of considering a discrete spatial domain and k-space and attempting to map the nonrectilinear sample into a rectilinear grid, the new formulation considers continuous spatial domain with a piecewise constant form representing the realistic display method of using pixels. The k-space is also assumed to be continuous with samples given by the available data. The problem is formulated as a linear system of equations that maps the acquired k-space points into the optimal pixel values that represent the image. Even though the linear system here is also large in size and dense, a simple transformation is utilized to convert the system into a sparse form that promises much smaller computational and storage efforts. The sparse system is then solved using the conjugate gradient iterative technique, which enables a flexible control of the degree of accuracy versus the computation time. The theory of the new method is described and the preliminary results of applying the technique to reconstruct images of a numerical phantom as well as real data from a spiral imaging sequence are presented.

2. THEORY

Consider $f(x,y)$ as the continuous-space spatial domain intensity distribution and let the available k-space (frequency domain) samples be $F(kx_i, ky_i)$, where i=0,1,…,L-1, and L is the number of samples. Given the method by which the image is usually displayed in practice, the spatial domain can be modeled as

piecewise constant function consisting of the sum of shifted gate-like functions representing the pixels of the image up to the desired resolution. That is,

$$f(x,y) = \sum_{n=0}^{N-1}\sum_{m=0}^{M-1} \alpha_{n,m} \cdot \Pi(x - x_n, y - y_m) \tag{1}$$

Consequently, the continuous Fourier transform of *f(x,y)* can be obtained as,

$$F(k_x,k_y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sum_{n=0}^{N-1}\sum_{m=0}^{M-1} \alpha_{n,m} \cdot \Pi(x - x_n, y - y_m) \cdot e^{-j2\pi(k_x x + k_y y)} dxdy, \tag{2}$$

which reduces to,

$$F(k_x,k_y) = Sinc(w_x k_x) \cdot Sinc(w_y k_y) \sum_{n=0}^{N-1}\sum_{m=0}^{M-1} \alpha_{n,m} \cdot e^{-j2\pi(k_x x_n + k_y y_m)}. \tag{3}$$

Here w_x and w_y correspond to the half of the pixel width in x and y directions. Hence, given the available arbitrarily located samples in the k-space, we can express the above equation as a linear system whose equations are obtained by substituting the set of L value for k_x and k_y. Hence, the linear system matrix has dimensions of L rows and M×N columns. The right-hand side vector of this linear system consists of the values of the acquired frequency samples indexed by their *i* value. Given this index and from the known locations of the samples in the k-space, the system matrix can be computed for a given arrangement of the pixel values in the *v* vector. The unknowns in this linear system are the pixel intensity values.

In order to simplify the practical implementation of the solution, we observe that multiplication of the discrete Fourier transform matrix **H,** whose inverse is simply its Hermitian, by the rows of the matrix results in compacting the energy within the row into only a small number of points instead of the original dense (as opposed to sparse) format. In order to preserve the linear system unchanged while taking advantage of this property, we multiply by the Fourier transformation matrix and its inverse such that,

$$\vec{b} = \mathbf{A}\vec{v} = \mathbf{A} \cdot \mathbf{H}^H \cdot \mathbf{H} \cdot \vec{v} = (\mathbf{H} \cdot \mathbf{A}^H)^H \cdot \vec{V} = \mathbf{M} \cdot \vec{V}, \tag{4}$$

where $\vec{V}$ is the 1-D discrete Fourier transform of the 1-D listing of pixel values and the matrix **M** is has most of the energy in its rows compacted into only a few elements (yet not sparse). In order to convert the matrix **M** into a sparse matrix, we utilize a compression strategy that allows the user to choose a percentage of the energy of the matrix operator to preserve. In particular, in each row, the element values are sorted by absolute value and only the first few row elements needed to reach or exceed this level are preserved. Once this process is done, the matrix is stored in a sparse matrix format and the conjugate gradient iteration is utilized to solve the linear system.

3. METHODS

3.1 Energy compacting transformation

The energy compacting transformation is implemented by Fourier transforming the rows of the linear system matrix **A**, which describes the particular mapping that corresponds to the k-space trajectory or sampling scheme used in the problem at hand. This outcome of this transformation is sorted by magnitude and only the largest few elements are selected such that their energy is above a certain predetermined percentage of the total energy of the kernel represented by this particular row. Subsequently, such elements are stored by their row and column positions and values as a part of the new sparse linear system matrix. Given that this process is done in a row-by-row fashion, the storage demands for this method are not very high since the huge system matrix need not be constructed or stored.

3.2 Sparse matrix manipulation

The outcome of the energy compacting step is a description of the linear system matrix as a sparse matrix with only a few nonzero elements in each row. The storage of such system can be performed using several techniques. Among the most efficient ways to do that is the row-indexed storage method[17], which requires only twice the size of the nonzero elements for their storage. Once this representation is done, matrix-vector multiplication operations are only equal in complexity to the number of nonzero elements, which is always computationally feasible. Also, Hermitian operations are also rather simple and do not generally pose a computational burden.

3.3 Conjugate gradient iterative solution

The method of conjugate gradient optimizes the solution of a linear system by removing the error components in a number of directions that span the space of the solution[15]. It has a number of advantages as a result of its unique scheme. The first is that the number of iterations to reach the solution has an upper bound of the dimension of the space of the linear system solution. Moreover, only a few iterations are usually required to reach a good accuracy for the solution. Another advantage is that the solution accuracy can be traded off with computation time rather flexibly. This allows the method to be customized for the particular application at hand by selecting a predetermined value for the number of iterations that correspond to the desired computation time. If accuracy is desired, an efficient implementation of this method may rely on a measure of the solution update in such a way that the stopping criterion is to have an update that is insignificant compared to the present solution. This can be described mathematically in terms of an arbitrary definition of vector norms. Given the sparse matrix format of the system matrix, the computational complexity of each of the conjugate gradient iterations is still in the order of the total number of nonzero elements in the sparse system matrix. This is quite reasonable and can be guaranteed to be equal to or lower than the computational effort of conventional gridding techniques. A description of the conjugate gradient iteration used in this work is provided in Appendix I.

4. RESULTS AND DISCUSSION

The proposed method was implemented to construct images from numerical phantoms as well as from actual magnetic resonance imaging data acquired from a spiral imaging sequence. The numerical phantom data were constructed from polar sampling of the analytical expression of the Shepp-Logan phantom[14]. The actual data were acquired on a 4T whole-body MRI scanner using a spiral sequence. The spiral sequence had 4 interleaves with 1836 points each to sample a 128×128 k-space matrix. The actual k-space was measured for this sequence to guarantee the accuracy of k-space sampling locations[18]. The k-space sampling information for both the numerical and actual data are used to construct the sparse system matrix which is stored as the reconstruction table and retrieved for online reconstruction using the conjugate gradient method.

In Figure 1, an illustration of the energy compacting transformation is presented. A system matrix was constructed for a 64×64 gridding problem under polar sampling as shown in (a). The resultant system matrix, which has a size of $64^2 \times 64^2$, is shown in (b). As can be seen, only a few elements are nonzero as opposed to the dense format prior to this transformation. The average number of nonzero elements per row for this particular case was 3 to achieve approximately 92% of the kernel energy, which is considered a rather efficient formulation.

The result of reconstructing the image of the actual data is shown in Figure 2 (left) and compared to the image obtained with the best conventional gridding reconstruction (right). The number of nonzero elements in the sparse linear system matrix was 81512, which is equal to 4.9 N^2, where N here is 128, to preserve an average of more than 92% of the energy of the row kernels. The number of conjugate gradient iterations used here was 5. As can be observed, the quality of the image reconstructed using the new technique is quite comparable to the results of conventional gridding.

The results of reconstructing the numerical Shepp-Logan phantoms are shown in Figures 3 and 4. In Figure 3, the results for an image size of 128×128 is provided where the results of all iterations are shown starting from the top left (N_{iter}=1) to the bottom right (N_{iter}=20). The system matrix in this case had 17192 nonzero elements or 10.9 N^2. The same is repeated for image size of 256×256 where the number of nonzero elements here is 293518 or 4.5 N^2. As can be seen, the iteration reaches a good solution after only a few steps.

It should be noted that the size and location of the large matrix elements that are selected to achieve the kernel energy percentage varies from one row to another. This means that the kernel used to perform the mapping is spatially varying. We notice also that the truncation using this method is optimal in the least-squares sense. The selection using this method avoids having the user choose more specific parameters of reconstruction like window size or neighborhood definition like other techniques[3,10]. Instead, a single parameter is used as a quality measure (the energy percentage) while another is used as a computation time

control (number of conjugate gradient iterations). Given that these parameters are more obvious to the MRI system user, the proposed method should fit well into clinical settings.

The computational complexity of the proposed system can be shown to be $O(N^2)$ to obtain the solution to the linear system. Then, to compute the image, an N^2-point FFT is required afterwards. This computational complexity is well within the range for conventional gridding methods as well as new gridding techniques based on SVD. The ability of the user to control the construction time is a unique feature in this iteration. This allows a quick, almost real-time computation of images for fast viewing by the radiologist. Also, it allows the radiologist to increase the accuracy of the reconstruction of a selected image at will simply by allowing additional iterations to run. Given the low complexity of iterations, such process can be performed during image viewing using console control just like zooming or gamma curve selection with virtually no noticeable delay. This is an obvious advantage of the new method.

5. CONCLUSIONS

A new method for gridding nonuniformly sampled frequency domain data is presented. This method provides the means for solving the reconstruction problem iteratively based on a practical model. The unique aspect of this technique is that the iteration includes an explicit reconstruction error measures that can be monitored and controlled for customized performance. More work is needed to investigate the use of the new method in practical clinical applications.

ACKNOWLEDGEMENTS

The author would like to thank Drs. X. Hu and S. Sarkar for providing the spiral data and Dr. A.S. Fahmy for his Shepp-Logan numerical phantom programs.

APPENDIX I: ALGORITHM FOR CONJUGATE GRADIENT ITERATION

The method of conjugate gradient describes a class of iterative techniques having the property of guaranteed convergence in a finite number of iterations[15,16]. Also, even when the system is ill conditioned, good estimates of the largest and smallest eigenvalues are not needed to determine the algorithm parameters. The basic idea of this method is to eliminate the residual error (i.e., the difference between the right-hand and left-hand sides of the linear system equation) along directions that are all mutually orthogonal with under transformation with the system matrix and spanning the space of the solution. The original formulation of this iteration requires the system to be real, square, symmetric and positive definite for the algorithm to work and provide a unique solution to the system[16]. Here, a modification of the technique is applied to complex Hermitian, positive semidefinite linear systems to compute the minimal least-squares solution[9]. That is, it is used to solve the normal equations of the system given the properties of the Grammian matrix. In particular, the conjugate gradient algorithm for solving the normal equation $\mathbf{A}^{\mathbf{H}}\mathbf{A}\vec{x} = \mathbf{A}\vec{b}$ is described as follows:

1. Set the initial solution $\vec{x}_0 = \vec{0}$.
2. Compute the initial residual $\vec{r}_0 = \vec{b} - \mathbf{A}\vec{x}_0$.
3. Compute first direction $\vec{p}_0 = \mathbf{A}^{\mathbf{H}}\vec{r}_0$.
4. Compute $c_m = \left\|\mathbf{A}^{\mathbf{H}}\vec{r}_m\right\|_2^2$, $d_m = \left\|\mathbf{A}\vec{p}_m\right\|_2^2$, $a_m = c_m / d_m$.
5. Update solution $\vec{x}_{m+1} = \vec{x}_m + a_m \cdot \vec{p}_m$, and update residual $\vec{r}_{m+1} = \vec{r}_m - a_m \cdot \mathbf{A}\vec{p}_m$.
6. Compute $e_m = \left\|\mathbf{A}^{\mathbf{H}}\vec{r}_{m+1}\right\|_2^2 \Big/ c_m$, and update direction $\vec{p}_{m+1} = \mathbf{A}^{\mathbf{H}}\vec{r}_{m+1} + e_m \cdot \vec{p}_m$.
7. Increement counter *m=m+1*, and repeat steps 4 through 6 until one of the following conditions is satisfied: $e_m = 0$, c_m is below a certain threshold, or the number of iterations reached a predetermined number N_{iter}.

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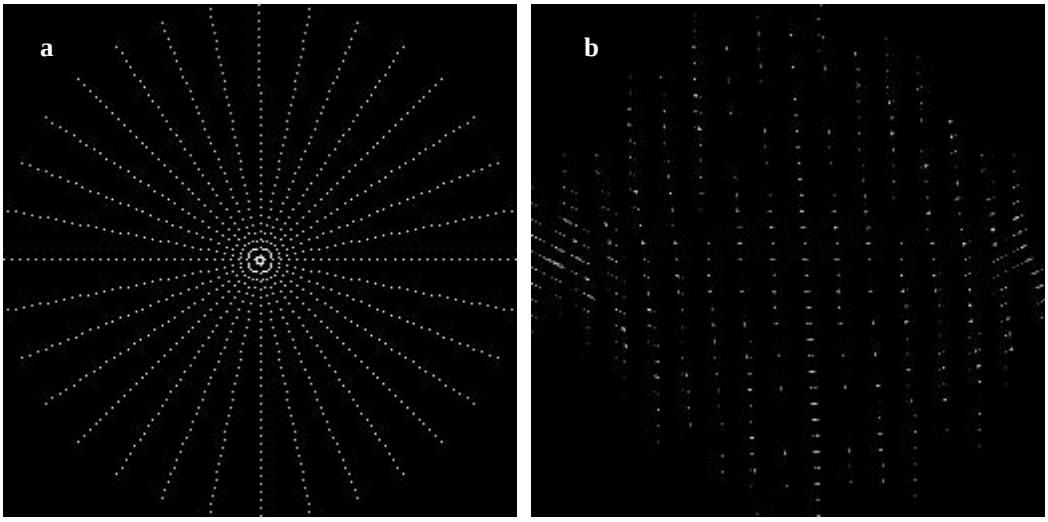

Figure 1. Illustration of the energy compacting transformation. (a) polar trajectory used and (b) the constructed system matrix after the energy compacting transformation with only a few nonzero elements.

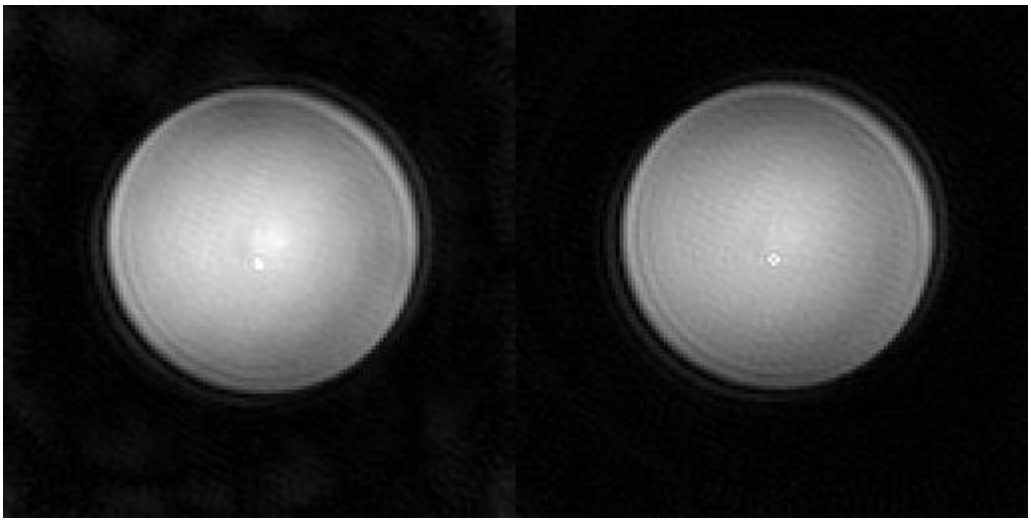

Figure 2. Images constructed with the new method (left) and the best image obtained with conventional gridding procedure (right).

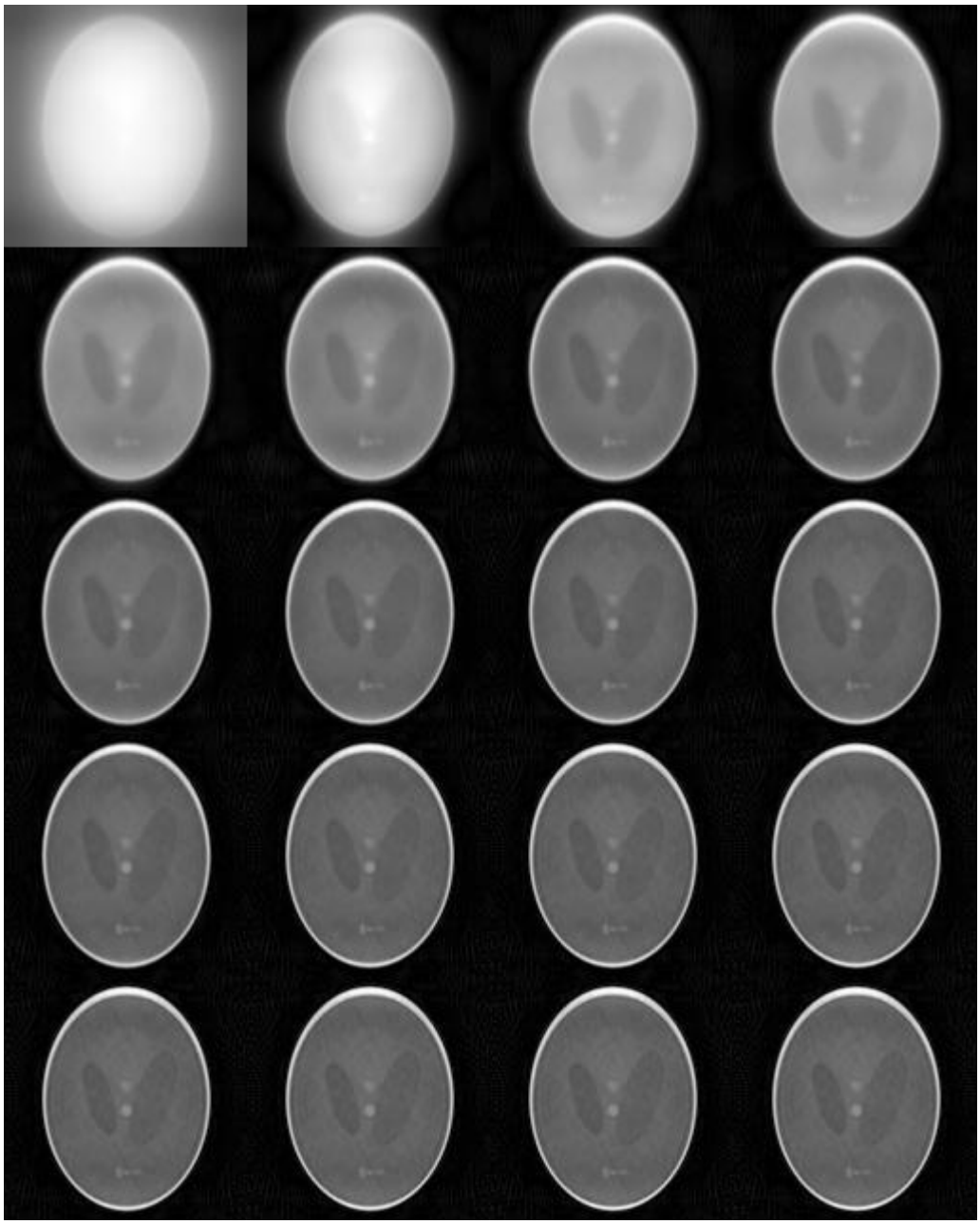

Figure 3. Reconstructed images from polar sampling of the analytical form of the Fourier transform of the Shepp-Logan in case when the image size is 128×128.

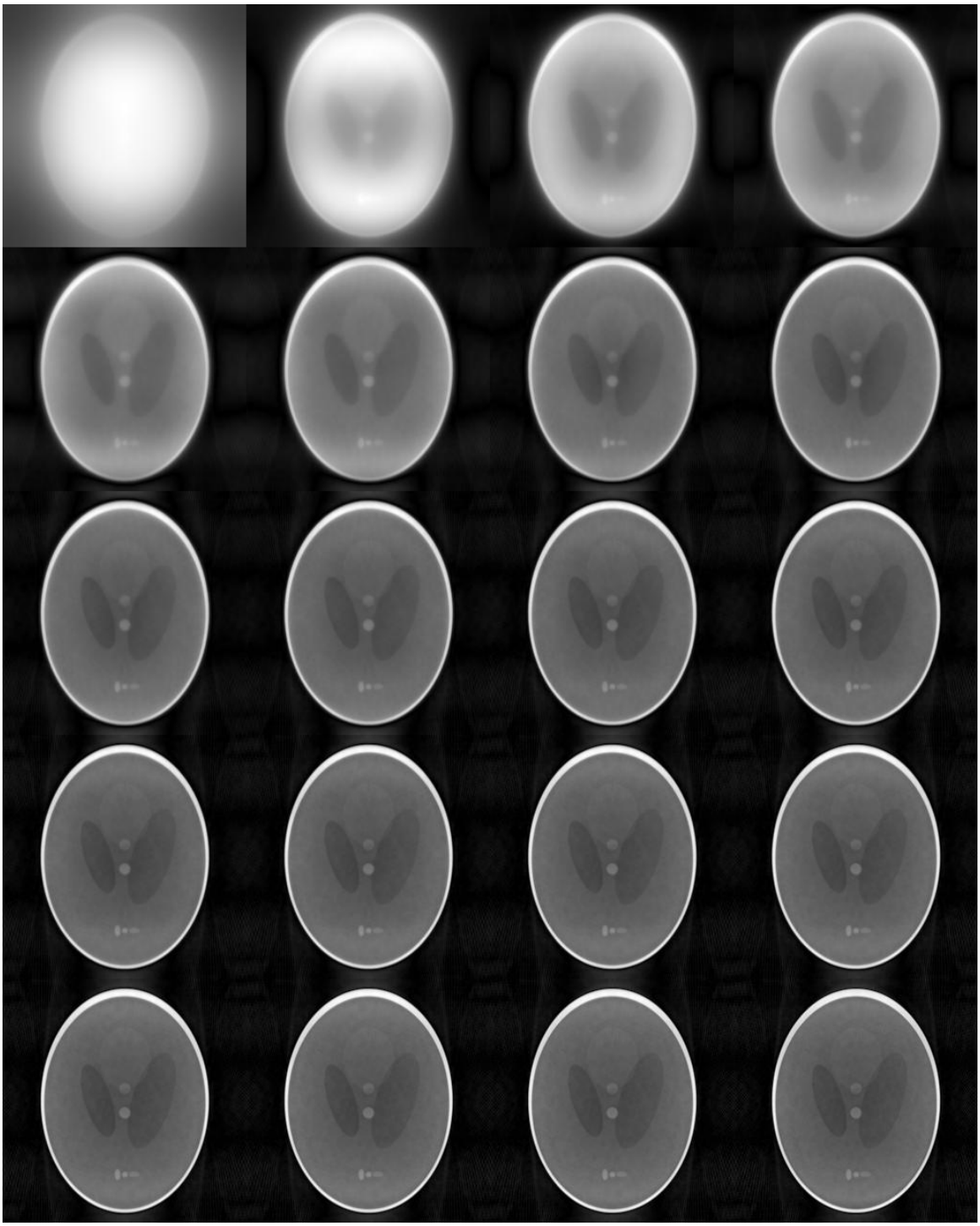

Figure 4. Reconstructed images from polar sampling of the analytical form of the Fourier transform of the Shepp-Logan in case when the image size is 256×256.